On the size of maximal binary codes with 2, 3, and 4 distances

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Research team

1. What is an s-distance set?
   A set with exactly s distinct nonzero distances.
   $\{x_1, x_2, \cdots, x_k\} \subset \mathbb{R}^n$ and $\{||x_i - x_j||\}_{i \neq j}$ has size s.
2. What spaces we consider in here?
   Hamming space $\mathcal{H}_2^n$: a n-dimensional vector space over $\mathbb{F}_2$.
   Johnson spaace $\mathcal{J}_2^{n,w}$: a collection of all vectors in $\mathcal{H}_2^n$ with w ones.
3. How do we define the distance?
   Hamming space: $d_{\mathcal{H}}(x, y)$ is the number of ones in $x - y$.
   Johnson space: $d_{\mathcal{J}}(x, y) = \frac{1}{2} d_{\mathcal{H}}(x, y)$.

Q : What is the maximum size of s-distance sets in the Hamming and Johnson spaces?

History of the computational method

Linear programming and semidefinite programming method were applied on coding theory.

1. 1973: Delsarte presented his linear programming method.
2. 1977: Larman, Rogers and Seidel(LRS) gave the integer conditions of 2-distance sets in $\mathbb{R}^n$ if $|X| > 2n+3$.
3. 2005: Schrijver used the Terwilliger algebra to give the semidefinite programming bounds improving $A(n,d)$.
4. 2011: Nozaki generalized the LRS theorem to s-distance sets in $\mathbb{R}^n$.
5. Barg-Musin and Musin-Nozaki applied the linear programming method and the generalized LRS theorem on s-distance sets in the Hamming and Johnson space.

Constructions

Proposition 1 (Barg-Musin)

For $2s \le n$, we have

$$A(\mathcal{H}_2^n, s) \ge \sum_{i=0}^{\lfloor \frac{s}{2} \rfloor} \binom{n}{s-2i}.$$

Example 1

If $s = 2$,

$$A(\mathcal{H}_2^n, 2) \ge \sum_{i=0}^{1} \binom{n}{2-2i} = 1 + \binom{n}{2}.$$

If $s = 3$,

$$A(\mathcal{H}_2^n, 3) \ge \binom{n}{1} + \binom{n}{3}.$$

Constructions

Theorem 2 (Barg-Musin)

For $s \leq n - w$ we have

$$A(\mathcal{J}_2^{n,w}, s) \geq \binom{n-w+s}{s}.$$

Proof.

For $s \leq n - w$, consider the set of binary vectors in $\mathcal{J}_2^{n,w}$ with ones in the first $w - s$ coordinates. Clearly, it forms an s-distance set of size $\binom{n-w+s}{s}$ with distances $\{1, \ldots, s\}$. □

Computational results

Define $A(M,s)$ as the maximum size of an s-distance set in the space M.

Theorem 3 (Barg-Musin, 10')

For $6 \leq n \leq 74$ and $n = 78$ with the exceptions of $n = 47, 53, 59, 65, 70$ and 71, $A(\mathcal{H}_2^n, 2) = 1 + \binom{n}{2}$.

By the LP and SDP method together with LRS theorem, we have the following results.

Theorem 4 (SDP bounds)

1. *For $6 \leq n \leq 74$ and $n = 78$ with the exceptions of $n = 70$ and 71, $A(\mathcal{H}_2^n, 2) = 1 + \binom{n}{2}$.*
2. $A(\mathcal{J}^{n,w}, 2) = \binom{n-w+2}{2}$ *for $35 \leq n \leq 100$, $w = 3, 4, 5, 6$.*

Conjectures

Inspired by our computational results, we start to ask the following questions.

1. Is $A(\mathcal{H}_2^n, 2) = 1 + \binom{n}{2}$ for $n \geq 6$?
2. Is $A(\mathcal{J}^{n,w}, 2) = \binom{n-w+2}{2}$ for $n \geq 3(w-1)$, $w = 3, 4, 5$ and 6 for $n \geq 35$?

To answer those questions, we need to develop a theoretic method which do not rely on a computer.

2-distance sets in the Hamming space

Spherical embedding

1. Consider the map f from the Hamming space $\mathcal{H}_2^n$ into the unit sphere S^{n-1} by sending $x = (x_1, \ldots, x_n)$ to $f(x) = (v_1, \ldots, v_n)$, where $v_i = \frac{1}{\sqrt{n}}(-1)^{x_i}$.
2. This map sends two vectors with distance d into two vectors in the unit sphere with the Euclidean inner product $1 - \frac{2d}{n}$.
3. We can regard a 2-distance set in the Hamming space with distances a, b as a spherical 2-distance set with inner product $\alpha = 1 - \frac{2a}{n}$ and $\beta = 1 - \frac{2b}{n}$.

For a spherical two-distance set in $\mathbb{R}^n$, we can construct an equiangular set in $\mathbb{R}^{n+1}$ with inner product

$$\frac{1}{\gamma} = \frac{\beta - \alpha}{2 - (\alpha + \beta)}$$

if $-1 \leq \alpha < \beta \leq 1$ and $\alpha + \beta < 0$.
The case $\alpha + \beta \geq 0$ is done by Musin who proved upper bound at most $1 + \binom{n}{2}$.
Our problem becomes to evaluate the upper bound of an equiangular set.

the upper bound of an equiangular set

Theorem 5 (Neumann)

$M_\alpha(n) \leq 2n,$ *unless* $1/\alpha$ *is an odd integer.*

Theorem 6 (Lemmens-Seidel)

$M_{\frac{1}{3}}(n) = 28$ *for* $7 \leq n \leq 15,$ *and* $M_{\frac{1}{3}}(n) = 2n - 2$ *for* $n \geq 15.$

Theorem 7 (Yu, 2017)

$M_{\frac{1}{a}}(n) \leq \frac{(a^2-2)(a^2-1)}{2}$ *for all* n *and* a *such that* $n \leq 3a^2 - 16$ *and* $a \geq 3.$

The upper bound of an equiangular set

Theorem 8 (Relative bound)

$M_\alpha(n) \leq \frac{n(1-\alpha^2)}{1-n\alpha^2}$, *for all* α *and* $n \in \mathbb{N}$ *such that* $n\alpha^2 < 1$.

Theorem 9 (Glazyrin-Yu)

$$M_{\frac{1}{a}}(n) \leq n\left(\frac{2}{3}a^2 + \frac{4}{7}\right) + 2, \text{ for all } a \geq 3 \text{ and for all } n \in \mathbb{N}.$$

Lemma 10

$M_{\frac{1}{a}}(n) \leq \binom{n-1}{2}$ *for* $n \geq 7$ *unless* a *is an odd integer and* $n \in \{a^2 - 1, a^2 - 2\}$.

The upper bound of an equiangular set

Proof.

i If a is not an odd integer, by Theorem 5 (Neumann), $M_{\frac{1}{a}}(n) \leq 2n \leq \binom{n-1}{2}$ for all $n \geq 7$. Thus, in the following, we only consider the case off odd integer a.

ii For $a^2 \leq n \leq 3a^2 - 16$, by Theorem 7 (Yu),

$$M_{\frac{1}{a}}(n) \leq \frac{(a^2-1)(a^2-2)}{2} \leq \binom{n-1}{2}.$$

□

Proof.

iii For $n \leq a^2 - 3$, by Theorem 8(Relative bounds),

$$M_{\frac{1}{a}}(n) \leq \frac{n(a^2-1)}{a^2-n} = n + \frac{n^2-n}{a^2-n} \leq n + \frac{n^2-n}{3} \leq \binom{n-1}{2}$$

for all $n \geq 13$. For $7 \leq n \leq 12$, we have $a \geq 5$. By Theorem 8 again,

$$M_{\frac{1}{a}}(n) \leq n + \frac{n^2-n}{a^2-n} \leq n + \frac{n^2-n}{13} \leq \binom{n-1}{2}.$$

□

The upper bound of an equiangular set

Proof.

iv For the last case, consider $n \geq 3a^2 - 15$. By Theorem 9 (G-Yu),

$$M_{\frac{1}{a}}(n) \leq n\left(\frac{2}{3}a^2 + \frac{4}{7}\right) + 2 \leq n\left(\frac{2}{3} \cdot \frac{n+15}{3} + \frac{4}{7}\right) + 2 \leq \binom{n-1}{2}$$

for all $n \geq 20$. For $12 \leq n \leq 19$ we have $a = 3$, and by Theorem 6

$$M_{\frac{1}{a}}(n) = 28 \leq \binom{n-1}{2}.$$

□

The upper bound for some spherical 2-distance sets

Lemma 11

Suppose $-1 \leq \alpha < \beta < 1$ *and* $\alpha + \beta < 0$. *Let* $\gamma = \frac{2-(\alpha+\beta)}{\beta-\alpha}$. *Then, for* $n \geq 6$,

$$g(n, \alpha, \beta) \leq \binom{n}{2}$$

unless γ *is an odd integer and* $n \in \{\gamma^2 - 2, \gamma^2 - 3\}$.

The upper bound of almost all 2-distance sets in $\mathcal{H}_2^n$

Theorem 12

For $n \geq 6$,

$$A(\mathcal{H}_2^n, a, b) \leq 1 + \binom{n}{2}$$

unless there is a positive integer m *such that* $a = (m+1)(2m+1)$, $b = m(2m+1)$ *and* $n \in \{(2m+1)^2 - 2, (2m+1)^2 - 3\}$.

Exceptional cases

Theorem 13 (Glazyrin-Yu)

Let X be a spherical two-distance set in $\mathbb{S}^{n-1}$ with scalar products α, β. Then

$$|X| \leq \frac{n+2}{1 - \frac{n-1}{n(1-\alpha)(1-\beta)}},$$

if the right hand side is positive.

Lemma 14

For all $n \geq 6$,

$$A(\mathcal{H}_2^n, a, b) \leq \binom{n}{2},$$

given that $a = (m+1)(2m+1)$, $b = m(2m+1)$ and $n \in \{(2m+1)^2 - 2, (2m+1)^2 - 3\}$ for some positive integer m.

Conclusion

Theorem 15

$A(\mathcal{H}_2^n, 2) = 1 + \binom{n}{2}$, *for* $n \geq 6$.

s-distance sets in the Johnson space

Revisit the linear programming method for the Johnson space

The linear programming method for the Johnson space

Theorem 16 (Delsarte)

Let $C \subset \mathcal{J}_2^{n,w}$ be a constant weight s-distance set with distances $\mathcal{D} = \{d_1, \ldots, d_s\}$. Then $|C| \leq LP_J(\mathcal{D})$, where

$$LP_J(\mathcal{D}) := \max \Big\{ \sum_{i=0}^{s} f_i : f_0 = 1; f_j \geq 0, j = 0, \ldots, s \\ \text{and } \sum_{j=0}^{s} f_j \psi_k(d_j) \geq 0, k = 0, 1, \ldots, w \Big\},$$

and

$$\psi_k(x) = \sum_{j=0}^{k} (-1)^j \frac{\binom{k}{j}\binom{n+1-k}{j}}{\binom{w}{j}\binom{n-w}{j}} \binom{x}{j}$$

are the Hahn polynomials.

Revisit the linear programming method

Theorem 17

Let $C \subset \mathcal{J}_2^{n,w}$ be a s-distance set with distance $\{d_1, \ldots, d_s\}$. Let $v_k = (\psi_k(d_1), \ldots, \psi_k(d_s))^\top$ for $k = 0, \ldots, w$. Suppose

1. $(-1)^s \begin{vmatrix} v_{k_1} & \cdots & v_{k_{j-1}} & v_0 & v_{k_{j+1}} & \cdots & v_{k_s} \end{vmatrix} \leq 0$ *for* $j = 1, \ldots, s$,
2. $(-1)^{s+1} \begin{vmatrix} v_{k_1} & \cdots & v_{k_s} \end{vmatrix} < 0$.

Then

$$|C| \leq \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & \psi_{k_1}(d_1) & \cdots & \psi_{k_s}(d_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \psi_{k_1}(d_s) & \cdots & \psi_{k_s}(d_s) \end{vmatrix}}{\begin{vmatrix} \psi_{k_1}(d_1) & \cdots & \psi_{k_s}(d_1) \\ \vdots & \ddots & \vdots \\ \psi_{k_1}(d_s) & \cdots & \psi_{k_s}(d_s) \end{vmatrix}}. \tag{1}$$

More results

Take $k_i = w + 1 - d_i$ for $i = 1, \ldots, s$ and apply Theorem 17.

Theorem 18

i $A(\mathcal{J}_2^{n,4}, 2) = \binom{n-2}{2}$ *for* $n \geq 9$.

ii $A(\mathcal{J}_2^{n,5}, 2) = \binom{n-3}{2}$ *for* $n \geq 12$.

iii $A(\mathcal{J}_2^{n,6}, 2) = \binom{n-4}{2}$ *for* $n \geq 35$.

iv $A(\mathcal{J}_2^{n,4}, 3) = \binom{n-1}{3}$ *for* $n \geq 11$.

v $A(\mathcal{J}_2^{n,5}, 3) = \binom{n-2}{3}$ *for* $n \geq 12$.

vi $A(\mathcal{J}_2^{n,6}, 3) = \binom{n-3}{3}$ *for* $n \geq 16$.

vii $A(\mathcal{J}_2^{n,5}, 4) = \binom{n-1}{3}$ *for* $n \geq 15$.

viii $A(\mathcal{J}_2^{n,6}, 4) = \binom{n-2}{3}$ *for* $n \geq 15$.

Conjecture: $A(\mathcal{J}_2^{n,w}, s) = \binom{n-w+s}{s}$ when n is large enough.

Open problems

- $A(S^{n-1}, 2) = \binom{n+1}{2}$ for $n = 46, 78...(2k+1)^2 - 3$?
- $A(\mathbb{R}^n, 2) =?$
  Maximum size of two-distance sets in Euclidean space is only known for $n \leq 8$.
- $A(\mathcal{H}_2^n, 3) = \binom{n}{1} + \binom{n}{3}$?

A Famous Spherical Design

In 2022, Maryna Viazovska received the Fields Medal for proving[1] that the E_8 root lattice is the densest packing in R^8.

The 240 vectors of E_8 root lattice form a tight spherical 7-design, which is crucial in many problems involving spherical codes.

[1] *The sphere packing problem in dimension 8.* Annals of Mathematics, 2017, 991-1015.

Definition:[2] For a finite subset X of the sphere S^{n-1} satisfies

$$\frac{1}{\sigma(S^{n-1})}\int_{S^{n-1}} f(x)d\sigma(x) = \frac{1}{|X|}\sum_{x\in X} f(x)$$

for all polynomial $f(x_1, ..., x_n)$ of degree at most t, we call X a spherical t-design.

Why do we consider the existence of spherical designs?

(i) It is a good subset that approximates the behavior of some functions over the whole sphere. (cubature formulas)

(ii) It is related to several problems involving spherical codes. (maximum separation codes, energy minimization, and kissing number problem)

[2]Delsarte, Goethals, & Seidel (1977), Spherical codes and designs. Geometry and Combinatorics, 68-93. Academic Press.

Existence of Spherical Designs

(i) DGS lower bound for spherical t-design $X \subset S^{n-1}$

$$|X| \geqslant \begin{cases} \binom{n+e-1}{e} + \binom{n+e-2}{e-1} & \text{if } t = 2e, \\ 2\binom{n+e-1}{e} & \text{if } t = 2e+1. \end{cases}$$

(The occurrence of equality is very rare.)

(ii) In 1984, Seymour and Zaslavsky[3] proved that for each $t \geqslant 1$ and $n \geqslant 1$, there exists a spherical t-design in S^n.

(iii) In 2013, Bondarenko, Radchenko, and Viazovska[4] proved that for each $t \geqslant 1$, $n \geqslant 1$ and $N \geqslant C_n t^n$ (for some constant C_n), there exists a spherical t-design in S^n consisting of N points.

[3] *Averaging sets: a generalization of mean values and spherical designs.* Advances in Mathematics 52.3, 213-240.

[4] *Optimal asymptotic bounds for spherical designs.* Annals of Mathematics, 443-452.

Linear Programming Bound

Delsarte's inequality: For any finite set $X \subset S^{n-1}$ and $k \geqslant 0$,

$$\sum_{(x,y)\in X^2} G_k^n(\langle x, y\rangle) \geqslant 0.$$

The equality holds for all $k = 1, ..., t$ iff X is a spherical t-design in S^{n-1}. This constraint can be transformed into a linear programming problem: [Delsarte-Goethals-Seidel, 1977] Let X be a spherical t-design in S^{n-1}. Suppose that $f : [-1, 1] \to \mathbb{R}$ is a polynomial with Gegenbauer expansion $f(x) = \sum_{k=0}^{d} f_k G_k^n(x)$ satisfying that

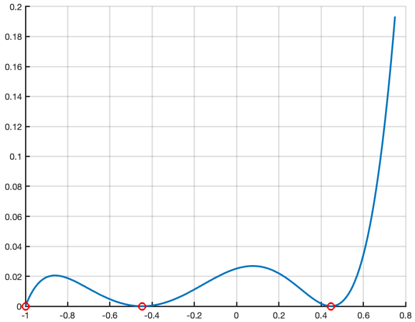
(i) $f(x) \geqslant 0$ for all $x \in [-1, 1]$,

(ii) $f_0 > 0$ and $f_k \leqslant 0$ for all $k = t + 1, ..., d$.

Then we have

$$|X| \geqslant \frac{f(1)}{f_0}.$$

Example: the 12 vertices of icosahedron



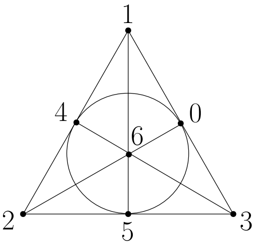
$$f = 1 + 2.2849 G_1^3 + 2.9499 G_2^3 + 2.7398 G_3^3 + 2.1281 G_4^3 + 1.0993 G_5^3 - 0.1242 G_7^3 - 0.0781 G_8^3.$$

By theorem, the size of spherical 5-design X in S^2 is $|X| \geqslant f(1) = 12$, which is attained by the icosahedron.

Definition: Let $V = \{1, 2, ..., v\}$ and $\mathfrak{B}$ be a collection of k-subset in V. The pair $(V, \mathfrak{B})$ is called a t-(v, k, λ) designs if there exists a constant λ such that for any t-subset $T \subset V$, the following holds:

$$|\{B \in \mathfrak{B} \mid T \subset B\}| = \lambda.$$

Example: the 2-(7,3,1) design (Fano plane)



$\{1, 2, 4\}$ $\{3, 4, 6\}$

$\{2, 3, 5\}$ $\{1, 5, 6\}$

$\{0, 1, 3\}$ $\{0, 4, 5\}$

$\{0, 2, 6\}$

Wilson's lower bound: $|\mathfrak{B}| \geqslant \binom{v}{e}$ if $t = 2e$; $|\mathfrak{B}| \geqslant 2\binom{v-1}{e-1}$ if $t = 2e - 1$.

Orthogonal Arrays

Definition: An $N \times n$ array A with entries from $\{0, 1, ..., q-1\}$ is said to be an orthogonal array $OA(N, n, q, t)$ $(1 \leqslant t \leqslant n)$ if every $N \times t$ subarray of A contains each t-tuple based on $\{0, 1, ..., q-1\}$ exactly λ times as a row.

Example: $OA(8, 4, 2, 3)$

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$

Rao's lower bound: $N \geqslant \sum_{i=0}^{e} \binom{n}{i} (q-1)^i$ if $t = 2e$.

Delsarte unified the notion of designs.

sphere S^{n-1}	Johnson scheme $\binom{[v]}{m}$	Hamming scheme $\mathbb{F}_q^n$
spherical designs	combinatorial designs	orthogonal arrays
spherical t-designs	t-(n, m, λ)	$OA(N, n, q, t)$
DGS bound	Wilson's bound	Rao's bound
$\sum G_k^n(\langle x, y \rangle) \geqslant 0$	$\sum H_k^n(d_J(x, y)) \geqslant 0$	$\sum K_k^n(d_H(x, y)) \geqslant 0$

Therefore, we can calculate the linear programming bounds for each of them similarly.

Question

From the numerical result, we notice that the smallest size of the design is far from the previous bound (or linear programming bound) in certain dimensions n and strengths t.

Question: Can we increase the lower bounds of the designs?

Here, we will provide two methods.

Method I: Semidefinite Programming on Size

The semidefinite programming (SDP) is the following problem:

$$\begin{array}{ll} \textbf{minimize} & c^T x \\ \textbf{subject to} & F_0 + \sum_{k=1}^{m} x_k F_k \succeq 0, \end{array}$$

(i) In 2005, Schrijver derived[5] several matrices $R_k \succeq 0$ and improved the upper bound for the $A(n, d)$-codes in $\mathbb{F}_2^n$ through SDP.

(ii) In 2008, Bachoc and Vallentin gave[6] an alternative proof to the kissing number problems in R^4 by deriving a matrix-valued positive-definite function $\sum_{(x,y,z)\in X^3} S_k^n(\langle x, y\rangle, \langle y, z\rangle, \langle z, x\rangle) \succeq 0$.

[5] *New code upper bounds from the Terwilliger algebra and semidefinite programming*. IEEE Transactions on Information Theory, 51(8), 2859-2866.

[6] *New upper bounds for kissing numbers from semidefinite programming.* Journal of the American Mathematical Society, 21(3), 909-924.

Method II: Energy Minimization Problem (on Sphere)

Let $X \subset S^{n-1}$ be a finite set of N-point, and $f : [-1, 1) \to \mathbb{R}$ be a function which is called potential. The energy minimization problem is to ask for the minimum energy

$$\mathcal{E}_f(n, N) := \inf \left\{ \sum_{x,y \in X, x \neq y} f(\langle x, y \rangle) \,\middle|\, X \subset S^{n-1}, |X| = N \right\}$$

corresponding to the potential f.

Remark:

(i) The Riesz potential $f(\|x - y\|) = \frac{1}{\|x-y\|^s}$ has been studied extensively[7].

(ii) It can be used to prove optimal codes.

[7] Kuijlaars and Saff, (1998). *Asymptotics for minimal discrete energy on the sphere*. Transactions of the American Mathematical Society, 350(2), 523-538.

We set f to be the sum of Gegenbauer polynomials of degree up to t

$$f(x) = G_1^n(x) + \cdots + G_t^n(x)$$

as the potential function. The N-point code on S^{n-1} will always be non-negative due to the Delsarte's inequality $\sum_{(x,y)\in X^2} G_k^n(\langle x, y\rangle) \geqslant 0$.

Moreover, the energy function equals zero if and only if there exists a N-point spherical t-design in S^{n-1}. Therefore, the existence of spherical designs can be transformed into an energy minimization problem.

Remarks:

(i) There are also LP and SDP bound[8] for the energy minimization problem on the sphere.

(ii) This method can also be applied to Hamming and Johnson designs.

[8] Cohn and Woo, (2012). *Three-point bounds for energy minimization*. Journal of the American Mathematical Society, 25(4), 929-958.

Theorem 19 (Delsarte-Yudin's LP bound)

Let $f:[-1,1]\to\mathbb{R}$ be the potential function. Suppose $h:[-1,1]\to\mathbb{R}$ is a function with Gegenbauer expansion $h(t)=\sum_{k=0}^{d} h_k G_k^n(t)$ such that $f_k\geqslant 0$ for all $k\geqslant 1$, and $f(t)\geqslant h(t)$ on $[-1,1]$. Then we have the lower-bound

$$\mathcal{E}_f(n,N)\geqslant h_0N^2-h(1)N.$$

Moreover, the equality holds if and only if
(1) $f(t)=h(t)$ for all $t\in\{\langle x,y\rangle \mid x,y\in X, x\neq y\}$; and
(2) $f_k\sum_{(x,y)\in X^2} G_k^n(\langle x,y\rangle)=0$ for all $k\geqslant 1$.

In our case, these conditions can be achieved by taking $f=h=G_1^n+\cdots+G_t^n$, and the resulting bound becomes

$$\mathcal{E}_f^*(n,N)=\mathcal{E}_f(n,N)+tN=h_0N^2-Nh(1)+TN=0.$$

Our Work

(i) We prove the non-existence of several spherical designs by solving the corresponding energy minimization problem with potential $f(x) = G_1^n(x) + ... + G_t^n(x)$ through SDP.

(ii) We improve the lower bounds of Hamming designs by using Schrijver's SDP bound for Hamming schemes.

(iii) Both methods DO NOT give any improvement on the lower bound of block designs (t=2), which is still a hard problem to solve.

Some SDP Bounds for Designs

dim	DGS bound	LP bound	SDP bund
10	275	283	286
11	352	352	361
12	442	442	445

Table: Improvement on spherical 6-designs

dim	Rao bound	LP bound	SDP bound
8	93	112	128
9	130	192	256
10	176	320	512

Table: Improvement on Hamming 6-designs (orthogonal arrays)

Considering the energy minimization problem, we can also show the non-existence of $OA(320, 10, 2, 6)$ and $OA(6144, 14, 2, 10)$. This resolves an open question from Sloane's book.

Thanks for your listening!