Vertical sets in the first Heisenberg group

IBS-DIMAG workshop on combinatorics and geometric measure theory

Terry Harris

University of Wisconsin-Madison

July 18, 2024

Let $\pi_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be

$$\pi_\theta(x, t) = \left(x \cos\theta, t - \frac{1}{4}x^2 \sin 2\theta \right).$$

Let $\pi_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be

$$\pi_\theta(x,t) = \left(x\cos\theta, t - \frac{1}{4}x^2 \sin 2\theta \right).$$

Let $A \subseteq \mathbb{R}^2$ have $\dim_E A = t$ (Hausdorff dimension w.r.t Euclidean metric).

Let $\pi_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be

$$\pi_\theta(x, t) = \left(x \cos\theta, t - \frac{1}{4}x^2 \sin 2\theta \right).$$

Let $A \subseteq \mathbb{R}^2$ have $\dim_E A = t$ (Hausdorff dimension w.r.t Euclidean metric).

- Given $A \subseteq \mathbb{R}^2$, what is $\dim_E (\pi_\theta(A))$, for "most" $\theta \in [0, \pi)$?

Let $\pi_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be

$$\pi_\theta(x,t) = \left(x\cos\theta, t - \frac{1}{4}x^2 \sin 2\theta \right).$$

Let $A \subseteq \mathbb{R}^2$ have $\dim_E A = t$ (Hausdorff dimension w.r.t Euclidean metric).

- Given $A \subseteq \mathbb{R}^2$, what is $\dim_E(\pi_\theta(A))$, for "most" $\theta \in [0,\pi)$?
- What about $\dim_P(\pi_\theta(A))$ if $\dim_P A = t$ with parabolic metric

$$d_P((x,t),(x',t')) = |x-x'| + |t-t'|^{1/2}?$$

Let $\pi_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be

$$\pi_\theta(x,t) = \left(x\cos\theta, t - \frac{1}{4}x^2 \sin 2\theta \right).$$

Let $A \subseteq \mathbb{R}^2$ have $\dim_E A = t$ (Hausdorff dimension w.r.t Euclidean metric).

- Given $A \subseteq \mathbb{R}^2$, what is $\dim_E(\pi_\theta(A))$, for "most" $\theta \in [0,\pi)$?
- What about $\dim_P(\pi_\theta(A))$ if $\dim_P A = t$ with parabolic metric

$$d_P((x,t),(x',t')) = |x-x'| + |t-t'|^{1/2}?$$

- If $|t-t'| \ll |x-x'| \ll |t-t'|^{1/2}$ then $d_P(\pi_\theta(x,t), \pi_\theta(x',t'))$ is erratic.

Let $\pi_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be

$$\pi_\theta(x,t) = \left(x\cos\theta, t - \frac{1}{4}x^2 \sin 2\theta \right).$$

Let $A \subseteq \mathbb{R}^2$ have $\dim_E A = t$ (Hausdorff dimension w.r.t Euclidean metric).

- Given $A \subseteq \mathbb{R}^2$, what is $\dim_E(\pi_\theta(A))$, for "most" $\theta \in [0,\pi)$?
- What about $\dim_P(\pi_\theta(A))$ if $\dim_P A = t$ with parabolic metric

$$d_P((x,t),(x',t')) = |x-x'| + |t-t'|^{1/2}?$$

- If $|t-t'| \ll |x-x'| \ll |t-t'|^{1/2}$ then $d_P(\pi_\theta(x,t), \pi_\theta(x',t'))$ is erratic.
- Expect that $\dim_P(\pi_\theta(A)) \geq \dim_P A$ a.e. θ.

- Problem is special case of conjecture of Balogh, Durand-Cartagena, Fässler, Mattila, Tyson about vertical projections $P_{\mathbb{V}_\theta^\perp} : \mathbb{H} \to \mathbb{R}^2$, where $\mathbb{H} = \mathbb{R}^3$. Special case is where $P_{\mathbb{V}_\theta^\perp}$ is restricted to $\{0\} \times \mathbb{R}^2$.

- Problem is special case of conjecture of Balogh, Durand-Cartagena, Fässler, Mattila, Tyson about vertical projections $P_{\mathbb{V}_\theta^\perp} : \mathbb{H} \to \mathbb{R}^2$, where $\mathbb{H} = \mathbb{R}^3$. Special case is where $P_{\mathbb{V}_\theta^\perp}$ is restricted to $\{0\} \times \mathbb{R}^2$.

- Conjecture is (for Heisenberg metric) $\dim P_{\mathbb{V}_\theta^\perp}(A) \geq \min\{\dim A, 3\}$, a.e. θ. In special case, $\dim = \dim_P$ and $P_{\mathbb{V}_\theta^\perp} = \pi_\theta$.

- Problem is special case of conjecture of Balogh, Durand-Cartagena, Fässler, Mattila, Tyson about vertical projections $P_{\mathbb{V}_\theta^\perp} : \mathbb{H} \to \mathbb{R}^2$, where $\mathbb{H} = \mathbb{R}^3$. Special case is where $P_{\mathbb{V}_\theta^\perp}$ is restricted to $\{0\} \times \mathbb{R}^2$.

- Conjecture is (for Heisenberg metric) $\dim P_{\mathbb{V}_\theta^\perp}(A) \geq \min\{\dim A, 3\}$, a.e. θ. In special case, $\dim = \dim_P$ and $P_{\mathbb{V}_\theta^\perp} = \pi_\theta$. (Cannot expect equality. Will discuss state of the art at end of the talk).

- Problem is special case of conjecture of Balogh, Durand-Cartagena, Fässler, Mattila, Tyson about vertical projections $P_{\mathbb{V}_\theta^\perp} : \mathbb{H} \to \mathbb{R}^2$, where $\mathbb{H} = \mathbb{R}^3$. Special case is where $P_{\mathbb{V}_\theta^\perp}$ is restricted to $\{0\} \times \mathbb{R}^2$.

- Conjecture is (for Heisenberg metric) $\dim P_{\mathbb{V}_\theta^\perp}(A) \geq \min\{\dim A, 3\}$, a.e. θ. In special case, $\dim = \dim_P$ and $P_{\mathbb{V}_\theta^\perp} = \pi_\theta$. (Cannot expect equality. Will discuss state of the art at end of the talk).

- B-DC-F-M-T considered special case where $A \subseteq \{0\} \times \mathbb{R}^2$, and showed

$$\dim_P(\pi_\theta(A)) \geq \frac{1}{2}(1 + \dim A), \quad 1 < \dim A < 2, \qquad \text{a.e. } \theta,$$

which improves over "classical" bound $\dim \pi_\theta(A) \geq \min\{1, \dim A\}$.

Theorem

(H. 24) (Conjecture is true for subsets of vertical planes) If $A \subseteq \mathbb{R}^2$ *is Borel, then* $\dim_P(\pi_\theta(A)) \geq \dim_P A$ *for a.e.* θ*.*

Theorem

(H. 24) (Conjecture is true for subsets of vertical planes) If $A \subseteq \mathbb{R}^2$ is Borel, then $\dim_P(\pi_\theta(A)) \geq \dim_P A$ for a.e. θ.

To understand idea of the proof, recall some ideas from (the Fourier analytic version of) Kaufman's proof of (Euclidean) Marstrand projection theorem in the plane, where $\rho_\theta : \mathbb{R}^2 \to \mathbb{R}$ is $(x, y) \mapsto (x \cos\theta + y \sin\theta)$.

Theorem

(H. 24) (Conjecture is true for subsets of vertical planes) If $A \subseteq \mathbb{R}^2$ *is Borel, then* $\dim_P (\pi_\theta(A)) \geq \dim_P A$ *for a.e.* θ*.*

To understand idea of the proof, recall some ideas from (the Fourier analytic version of) Kaufman's proof of (Euclidean) Marstrand projection theorem in the plane, where $\rho_\theta : \mathbb{R}^2 \to \mathbb{R}$ is $(x, y) \mapsto (x \cos\theta + y \sin\theta)$.

- $\dim A > s$ iff $\exists \mu$ on A with $I(\mu) = \int (f_s * \mu)(x)\, d\mu(x) < \infty$, where $f_s(x) = |x|^{-s}$.

Theorem

(H. 24) (Conjecture is true for subsets of vertical planes) If $A \subseteq \mathbb{R}^2$ *is Borel, then* $\dim_P(\pi_\theta(A)) \geq \dim_P A$ *for a.e.* θ*.*

To understand idea of the proof, recall some ideas from (the Fourier analytic version of) Kaufman's proof of (Euclidean) Marstrand projection theorem in the plane, where $\rho_\theta : \mathbb{R}^2 \to \mathbb{R}$ is $(x, y) \mapsto (x \cos\theta + y \sin\theta)$.

- $\dim A > s$ iff $\exists \mu$ on A with $I(\mu) = \int (f_s * \mu)(x)\, d\mu(x) < \infty$, where $f_s(x) = |x|^{-s}$.
- If $f_s(x) = |x|^{-s}$ on $\mathbb{R}^n$ then $\widehat{f_s} = c f_{n-s}$.

Theorem

(H. 24) (Conjecture is true for subsets of vertical planes) If $A \subseteq \mathbb{R}^2$ is Borel, then $\dim_P(\pi_\theta(A)) \geq \dim_P A$ for a.e. θ.

To understand idea of the proof, recall some ideas from (the Fourier analytic version of) Kaufman's proof of (Euclidean) Marstrand projection theorem in the plane, where $\rho_\theta : \mathbb{R}^2 \to \mathbb{R}$ is $(x, y) \mapsto (x \cos\theta + y \sin\theta)$.

- $\dim A > s$ iff $\exists \mu$ on A with $I(\mu) = \int (f_s * \mu)(x)\, d\mu(x) < \infty$, where $f_s(x) = |x|^{-s}$.
- If $f_s(x) = |x|^{-s}$ on $\mathbb{R}^n$ then $\widehat{f_s} = cf_{n-s}$.
- If $I_s(\mu) < \infty$, by Plancherel and above,

$$\int_0^\pi I_s(\pi_{\theta\#}\mu)\, d\theta = c \int_0^\pi \int_{\mathbb{R}} r^{s-1} \left|\widehat{\mu}\,(r\cos\theta, r\sin\theta)\right|^2 dr\, d\theta = cI_s(\mu) < \infty.$$

Theorem

(H. 24) (Conjecture is true for subsets of vertical planes) If $A \subseteq \mathbb{R}^2$ is Borel, then $\dim_P(\pi_\theta(A)) \geq \dim_P A$ for a.e. θ.

To understand idea of the proof, recall some ideas from (the Fourier analytic version of) Kaufman's proof of (Euclidean) Marstrand projection theorem in the plane, where $\rho_\theta : \mathbb{R}^2 \to \mathbb{R}$ is $(x, y) \mapsto (x \cos\theta + y \sin\theta)$.

- $\dim A > s$ iff $\exists \mu$ on A with $I(\mu) = \int (f_s * \mu)(x)\, d\mu(x) < \infty$, where $f_s(x) = |x|^{-s}$.
- If $f_s(x) = |x|^{-s}$ on $\mathbb{R}^n$ then $\widehat{f_s} = c f_{n-s}$.
- If $I_s(\mu) < \infty$, by Plancherel and above,

$$\int_0^\pi I_s(\pi_{\theta\#}\mu)\, d\theta = c \int_0^\pi \int_{\mathbb{R}} r^{s-1} \left|\widehat{\mu}\,(r\cos\theta, r\sin\theta)\right|^2\, dr\, d\theta = cI_s(\mu) < \infty.$$

- Implies $\dim \rho_\theta(A) \geq \dim A$, a.e. $\theta \in [0, \pi)$.

To adapt this to parabolic metric:

To adapt this to parabolic metric:

- Replace parabolic metric $|x - x'| + |t - t'|^{1/2}$ with equivalent

$$\|(x,t) - (x',t')\|, \qquad \|(x,t)\| = \left(|x|^4 + t^2\right)^{1/4}.$$

To adapt this to parabolic metric:

- Replace parabolic metric $|x - x'| + |t - t'|^{1/2}$ with equivalent

$$\|(x,t) - (x',t')\|, \qquad \|(x,t)\| = \left(|x|^4 + t^2\right)^{1/4}.$$

- Define $f_s(x) = \|x\|^{-s}$. Then for $1 < s < 3$,

$$\widehat{f_s} \lesssim f_{3-s}.$$

To adapt this to parabolic metric:

- Replace parabolic metric $|x - x'| + |t - t'|^{1/2}$ with equivalent

$$\|(x,t) - (x',t')\|, \qquad \|(x,t)\| = \left(|x|^4 + t^2\right)^{1/4}.$$

- Define $f_s(x) = \|x\|^{-s}$. Then for $1 < s < 3$,

$$\widehat{f_s} \lesssim f_{3-s}.$$

- This step relies on Basset's integral formula

$$K_\nu(z) = \frac{(2z)^\nu \Gamma\left(\nu + \frac{1}{2}\right)}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-iu}(z^2 + u^2)^{-\nu - \frac{1}{2}}\, du.$$

To adapt this to parabolic metric:

- Replace parabolic metric $|x - x'| + |t - t'|^{1/2}$ with equivalent

$$\|(x,t) - (x',t')\|, \qquad \|(x,t)\| = \left(|x|^4 + t^2\right)^{1/4}.$$

- Define $f_s(x) = \|x\|^{-s}$. Then for $1 < s < 3$,

$$\widehat{f_s} \lesssim f_{3-s}.$$

- This step relies on Basset's integral formula

$$K_\nu(z) = \frac{(2z)^\nu \Gamma\left(\nu + \frac{1}{2}\right)}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-iu}(z^2 + u^2)^{-\nu-\frac{1}{2}}\, du.$$

- Apply Plancherel to $\int_0^\pi \int (f_s * \pi_{\theta\#}\mu)(x)\, d\pi_{\theta\#}\mu(x)$, reduces to

$$\iint \int_{[0,\pi)\setminus bad} \int_0^{2^j} \int_0^{2^{2j}} e\left(r(x-x')\cos\theta + \rho\left(t - t' - \frac{1}{4}(x^2 - x'^2)\sin 2\theta\right)\right) d\rho\, dr\, d\theta\, d\mu(x,t)\, d\mu(x',t') \ll 2^{j(3-s)}\mu(\mathbb{R}^2) c_{s+\epsilon}(\mu) \qquad \forall j.$$

- $c_\alpha(\mu)$ is upper density of μ. Unlike Kaufman, cannot use $I_\alpha(\mu)$ for this approach.

- $c_\alpha(\mu)$ is upper density of μ. Unlike Kaufman, cannot use $I_\alpha(\mu)$ for this approach.

- The estimate is proved via induction on the scale 2^j.

- In general case, best know a.e. lower bound for $\dim P_{\mathbb{V}_\theta^\perp}(A)$ is due to Fässler-Orponen, and is

$$\dim\left(P_{\mathbb{V}_\theta^\perp}(A)\right) \geq \begin{cases} \dim A & \dim A \leq 2 \\ 2 & 2 \leq \dim A \leq 5/2 \\ 2\dim A - 3 & 5/2 \leq \dim A \leq 3. \end{cases}$$

- In general case, best know a.e. lower bound for $\dim P_{\mathbb{V}_\theta^\perp}(A)$ is due to Fässler-Orponen, and is

$$\dim\left(P_{\mathbb{V}_\theta^\perp}(A)\right) \geq \begin{cases} \dim A & \dim A \leq 2 \\ 2 & 2 \leq \dim A \leq 5/2 \\ 2\dim A - 3 & 5/2 \leq \dim A \leq 3. \end{cases}$$

- Their argument for $\dim A > 5/2$ uses energies ($\dim A < 3$) or L^2 norms ($\dim A = 3$). An example of B-DC-F-M-T (the parabola example) shows that for $1 \leq \dim A \leq 2$, $\frac{1}{2} + \frac{\dim A}{2}$ is the best possible via the standard energy method.