Star clusters in independence complexes of hypergraphs

Jinha Kim

Chonnam National University
& IBS Discrete Mathematics Group

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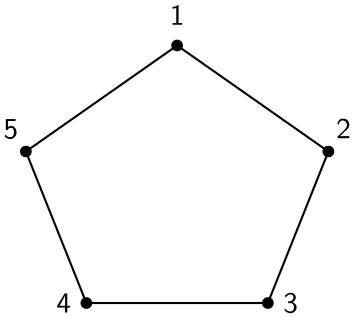
Independence complexes of graphs

- A **simplicial complex** on the ground set V is a collection of subsets of V such that it is closed under taking subsets.
- A **(simple) graph** G is a pair of a vertex set $V(G)$ and an edge set $E(G)$, where an edge $e \in E(G)$ is a vertex subset of size 2.
- $W \subseteq V(G)$ is an **independent set** of a graph G if W does not contain an edge of G as a subset.
- The **independence complex** of a graph G is

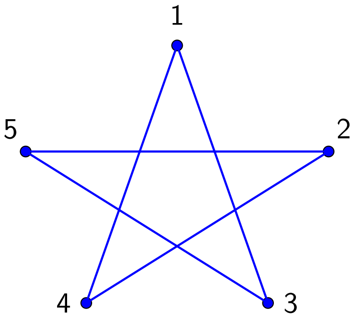
$$I(G) = \{W \subseteq V(G) : W \text{ is an independent set of } G\}.$$

- $\{$Independence complexes of graphs$\}= \{$Clique complexes of graphs$\}$ $= \{$Flag complexes$\}$.
- A **flag complex** is a simplicial complex where all minimal non-faces have size 2.

Independence complexes of cycles

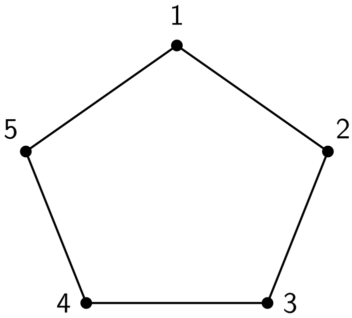


5-cycle C_5



$I(C_5) \simeq \mathbb{S}^1$

Independence complexes of cycles



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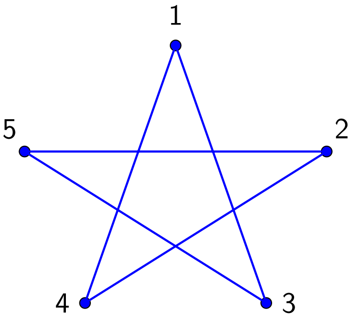

$I(C_5) \simeq \mathbb{S}^1$

Kozlov(1999):

$$I(C_\ell) \simeq \begin{cases} \mathbb{S}^k \vee \mathbb{S}^k & \text{if } \ell = 3k+3, \\ \mathbb{S}^k & \text{if } \ell = 3k+2 \text{ or } 3k+4. \end{cases}$$

Kalai–Meshulam conjecture

- A **ternary cycle** is a cycle of length divisible by 3.
- A graph G is **ternary** if it has no induced ternary cycles.

Conjecture 1 (Kalai–Meshulam)

For a ternary graph G, the number of independent sets of odd size and the number of independent sets of even size differ by at most 1.

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For a ternary graph G, the number of independent sets of odd size and the number of independent sets of even size differ by at most 1.

(the number of independent sets of odd size)-(the number of independent sets of even size) $=$ **(reduced) Euler characteristic of** $I(G)$$=\tilde{\chi}(I(G))$.

Conjecture 1 (reformulated) (Kalai–Meshulam)

For a ternary graph G, $|\tilde{\chi}(I(G))| \leq 1$.

Kalai–Meshulam conjecture

Conjecture 2 (Kalai–Meshulam)

$\sum_i \tilde{\beta}_i(I(G)) \leq 1$ for a ternary graph G.

Remark: $\tilde{\chi}(X) = \sum_i (-1)^i \tilde{\beta}_i(X)$ for a simplicial complex X.

$\implies |\tilde{\chi}(X)| = |\sum_i (-1)^i \tilde{\beta}_i(X)| \leq \sum_i \tilde{\beta}_i(X)$.

Conjecture 2 $\implies$ **Conjecture 1**.

The **total Betti number** of X is $\beta(X) = \sum_i \tilde{\beta}_i(X)$.

Kalai–Meshulam conjecture

Conjecture 3 (Engström)

For a ternary graph G, $I(G)$ is either contractible or homotopy equivalent to a sphere.

- $\tilde{\beta}_i(X) = 0$ for all i if X is contractible.
- For a d-dimensional sphere $\mathbb{S}^d$, $\tilde{\beta}_d(\mathbb{S}^d) = 1$ and $\tilde{\beta}_i(\mathbb{S}^d) = 0$ if $i \neq d$.

Conjecture 3 $\implies$ **Conjecture 2**.

Kalai–Meshulam Conjecture

- Chudnovsky–Scott–Seymour–Spirkl (2020): proved **Conjecture 1**.
- Zhang–Wu (2023): proved **Conjecture 2**.
- Engström (2020): proved a weaker version of **Conjecture 3**.

  "If a graph G has no ternary cycles (including non-induced ternary cycles), then $I(G)$ is either contractible or homotopy equivalent to a sphere."

Theorem 1 (K., 2022)

A graph G is ternary if and only if $I(H)$ is either contractible or homotopy equivalent to a sphere for every induced subgraph H.

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Theorem 1 (K., 2022)

A graph G is ternary if and only if $I(H)$ is either contractible or homotopy equivalent to a sphere for every induced subgraph H.

Q. Can we generalize such results for any simplicial complexes?

Independence complexes of hypergraphs

- A **hypergraph** H is a pair of a vertex set $V(H)$ and an edge set $E(H)$, where an edge $e \in E(G)$ is a non-empty subset of $V(H)$.
- $W \subseteq V(H)$ is an **independent set** of H if W does not contain an edge of H as a subset.
- The **independence complex** of H is

$$I(H) = \{W \subseteq V(H) : W \text{ is an independent set of } H\}.$$

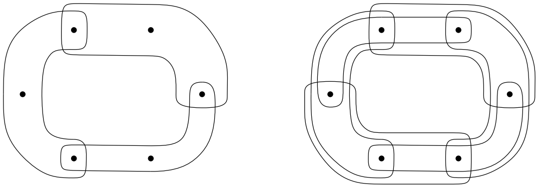
- Every simplicial complex is an independence complex of some hypergraph.

  For a simplicial complex X on V, let H_X be the hypergraph on V where the edges are the minimal non-faces of X. Then $X = I(H_X)$.

Independence complexes of hypergraphs

A **Berge cycle of length k** in H is given as $C = v_1 e_1 v_2 e_2 \dots v_k e_k$ with the following conditions.

1. $v_1, v_2, \dots, v_k$ are distinct vertices of H,
2. $e_1, e_2, \dots, e_k$ are distinct edges of H,
3. $v_i, v_{i+1} \in e_i$ for all $i \in [k]$, where $v_{k+1} = v_1$.



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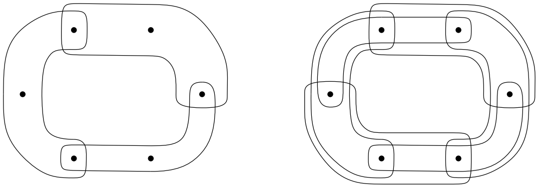

A **ternary Berge cycle** is a Berge cycle of length divisible by 3.

Theorem 2 (K., in preparation)

If a hypergraph H has no ternary Berge cycle, then $\beta(I(H)) \leq 1$.

Proof idea

Theorem 3 (K., in preparation)

For a hypergraph H has no ternary Berge cycle, there is a ternary graph G such that $\beta(I(H)) = \beta(I(G))$.

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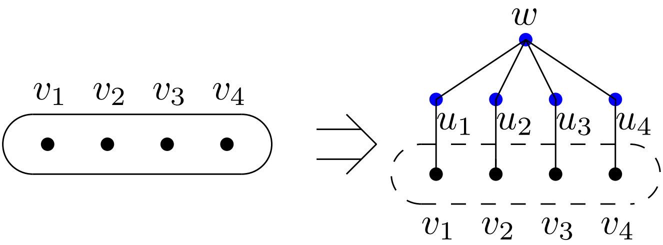


Let $e = \{v_1, v_2, \dots, v_k\}$ be an edge of H.
The **hypergraph H_e** obtained from H by deleting the edge e and adding new vertices $\{w, u_1, u_2, \dots, u_k\}$ and new edges

$$\{\{w, u_1\}, \{w, u_2\}, \dots, \{w, u_k\}, \{u_1, v_1\}, \{u_2, v_2\}, \dots, \{u_k, v_k\}\}.$$

Proof idea

Lemma 1 (K., in preparation)

For a hypergraph H and an edge e of H, $I(H_e) \simeq \Sigma I(H)$.

$$\implies \beta(I(H_e)) = \beta(I(H)).$$

Lemma 2 (K., in preparation)

For a hypergraph H and an edge e of H, if H has no ternary Berge cycle, then neither does H_e.

Star cluster in independence complexes

Theorem (Barmak, 2013)

Let G be a graph and v be a vertex that is not isolated and not contained in a triangle. Then

$$I(G) \simeq \Sigma(\mathrm{st}(v) \cap SC(N(v))).$$

Corollary: If v is not isolated and not contained in a triangle, then

$I(G) \simeq \Sigma X$ for some simplicial complex X on $V(G) \setminus \{v\}$.

Star cluster in independence complexes

Theorem 4 (K., in preparation)

Let H be a hypergraph and v be a vertex that is not isolated and not contained in a Berge cycle of length 3. Then

$$I(H) \simeq \Sigma(\mathsf{st}(v) \cap (\bigcup_{i=1}^{k} \mathsf{st}(\tilde{e}_i))),$$

where $e_1, e_2, \ldots, e_k$ are the edges of H containing v and $\tilde{e}_i = e_i \setminus \{v\}$.

Corollary: If v is not isolated and not contained in a Berge cycle of length 3, then

$$I(H) \simeq \Sigma I(H_v) \text{ for some hypergraph } H_v \text{ on } V(H) \setminus \{v\}.$$

Star clusters in independence complexes

Corollary: If v is not isolated and not contained in a Berge cycle of length 3, then $I(H) \simeq \Sigma I(H_v)$ for some hypergraph H_v on $V(H) \setminus \{v\}$.

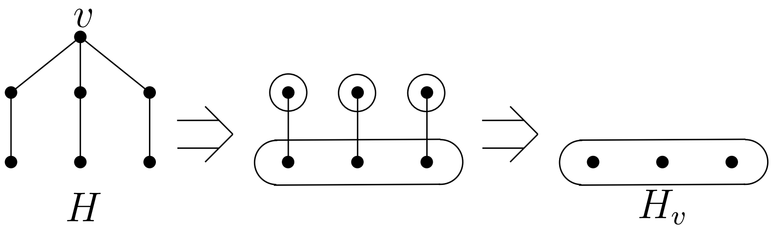
Let $e_1, \ldots, e_k$ be the edges of H containing v and $\tilde{e_i} = e_i \setminus \{v\}$.
Then $\tilde{e_1}, \tilde{e_2}, \ldots, \tilde{e_k} \in H_v$ and all other edges of H_v are given as follows.

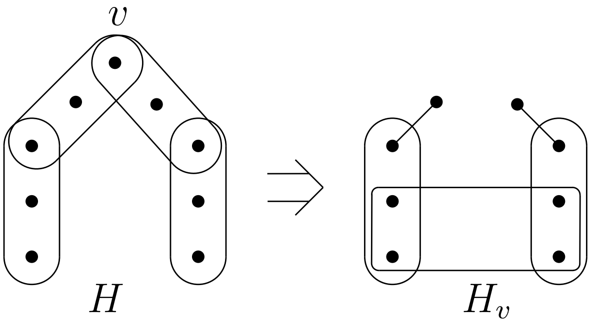
Theorem 5 (K., in preparation)

$f \in H_v \setminus \{\tilde{e_1}, \tilde{e_2}, \ldots, \tilde{e_k}\}$ if and only if f satisfies the following:

1. $\exists h_i \in H \setminus \{e_1, e_2, \ldots, e_k\}$ s.t. $h_i \subseteq f \cup e_i$ for all $i \in [k]$.
2. f is inclusion-wise minimal subject to (1).

Star clusters in independence complexes





Future research

Q. If H has no ternary Berge cycle, then $I(H)$ is either contractible or homotopy equivalent to a sphere?

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Q. Any other applications of star clusters in independence complexes of hypergraphs?

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Thank you!