# Falconer-type problems for dot products

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Background
- Discrete viewpoint
- Fractal setting

Results
- Distances
- Dot products

# Notation

Asymptotics. Let $X$ and $Y$ depend on an integer parameter $n$.

- We write $X \lesssim Y$ when there exists a constant $C$, independent of $n$, such that $X \leq CY$, for all sufficiently large $n$, or $X = O(Y)$.

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- With the symbol $\lessapprox$, we are typically burying logarithmic factors.

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- See Brass, Moser, Pach for more.

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$$u_3(n) \lesssim n^{\frac{4}{3}}.$$

- **Thm:** (Zahl, 2017) For any $\epsilon > 0$,

$$u_3(n) \lesssim n^{\frac{295}{197}+\epsilon}.$$

# Unit distance problem for $d \geq 4$

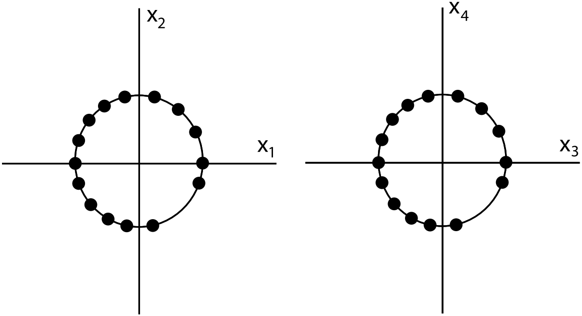


Figure: In dimensions 4 and up, there can be $\gtrsim n^2$ unit distances. Counterexample due to Lenz: $n/2$ points on the unit circle in the first two dimensions, the rest on a unit circle in the next two dimensions. Note that this is a "low-dimensional" set in higher dimensions.

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- This gives a sharp bound of $n^{\frac{4}{3}}$ for $n$ points in the plane.
- Simple ("low-dimensional") construction shows the bound is $n^2$ in dimensions three and higher.

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- This was proved by Guth and Katz in 2010.

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- **Thm:**(Hanson, Roche-Newton, S., 2021) Improved the exponent to $\frac{2}{3} + \frac{1}{2739}$.

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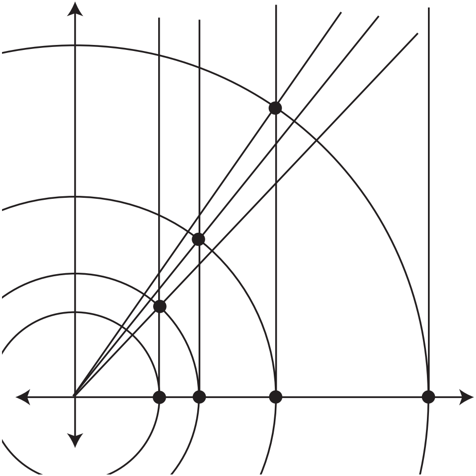
- Rotate and scale so that $(1, 0)$ is in our set.

# Devil's dartboard sketch (cont.)



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- $$\Pi(P) = \{(a, sa) \cdot (a', s'a')\} = \{aa' + ss'aa'\} = AA(1 + SS)$$

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- Note, if $|BB| \sim |B|$, then $B$ is "like" a geometric progression.
- So if $n^{\frac{2}{3}}$ is sharp, then $(1+SS)$ behaves like a geometric progression.
- The crux is showing that $(1+SS)$ cannot behave like a geometric progression, using Plünnecke, Garaev-Shen, Rudnev-Stevens, and "Solymosi squeeze" for convex sets.

# Quick note about finite fields/rings

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- ...meaning that each dot product has essential equal representation...
- ...but we have no idea how to exploit this.

# Falconer distance problem

- Given a compact subset $E \subset \mathbb{R}^d$, define $\Delta(E)$ to be the set of distances determined by pairs of points in $E$, that is:

$$\Delta(E) = \{|x - y| : x, y \in E\}.$$

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- Falconer proved $s > \frac{d+1}{2}$ initially. In the plane, the current record is $s > \frac{5}{4}$, due to Guth, Iosevich, Ou, and Wang. Higher dimensional results in various papers by these authors and Du, Ren, Wilson, and Zhang.

# Falconer's estimate

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- In the paper introducing his eponymous distance problem (How large of a Hausdorff dimension guarantees a positive measure of distinct distances?), Falconer proved that if $\dim_{\mathcal{H}} \text{supp}(\mu) = s > \frac{d+1}{2}$, then for any $\epsilon > 0$,

$$I_s(\mu) < \infty \Rightarrow (\mu \times \mu)\{(x, y) : 1 \leq |x - y| \leq 1 + \epsilon\} \lesssim \epsilon.$$

# Euclidean distance

## Theorem (Mattila, 1987)

*When $d = 2$, there exists a measure $\mu$ that will fail the analog of Falconer's estimate for $s < \frac{d+1}{2}$.*

- For $s < \frac{d+1}{2}$,

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- Cartesian product of a Cantor set and an interval.
- Extended to $d = 3$ by Iosevich, and S. in 2010.
- Unclear how to extend to higher dimensions.

# Mattila-type construction for $d = 2, 3$

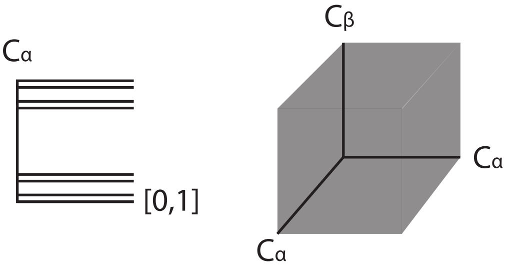


Figure: In dimension 2, we have $[0,1] \times \mathcal{C}_\alpha$. In dimension 3, we have $\mathcal{C}_\alpha \times \mathcal{C}_\alpha \times \mathcal{C}_\beta$.

# Non-Euclidean distance

Theorem (Iosevich, S., 2010–2016)

*There exists a centrally symmetric convex body B with smooth boundary and non vanishing curvature and a measure $\mu$ such that distances measured by dilates of B will fail the analog of Falconer's estimate for $s < \frac{d+1}{2}$.*

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- Discrete-fractal conversion mechanism - Hofmann, Iosevich, Jorati, Łaba, Uriarte-Tuero.

# Valtr's construction

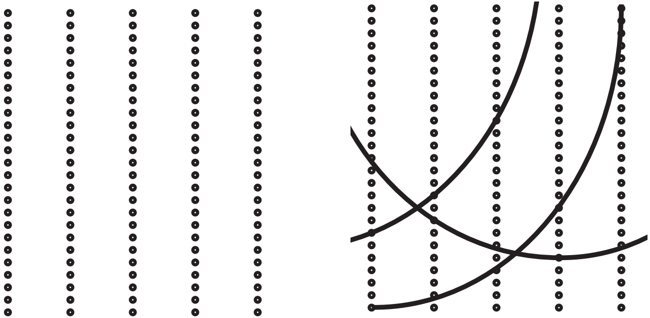

Figure: For stage $n$, set $m = n^{\frac{1}{3}}$. The points have coordinates $(\frac{i}{m}, \frac{j}{m^2})$, for $i = 1 \ldots m$ and $j = 1 \ldots m^2$. The "circles" are parabolic arcs glued together. The limit of this construction will be the support of the measure $\mu$.

# Falconer-type dot product problem

- Given a compact subset $E \subset \mathbb{R}^d$, recall $\Pi(E)$ is the set of dot products determined by pairs of points in $E$, that is:

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- Partial results ($s > \frac{d+1}{2}$): Eswarathasan, Iosevich, Palsson, Taylor, Uriarte-Tuero, etc., avoiding fractal devil's dartboard.

# Dot products in two dimensions

Theorem (Eswarathasan, Iosevich, Taylor, 2010)

*For any $s < \frac{3}{2}$, there exists a measure $\mu$ on $[0,1]^2$ with $\dim_{\mathcal{H}} supp(\mu) = s$, that will fail the analog of Falconer's estimate for dot products.*

- In $[0,1]^2$, for $s < \frac{3}{2}$,

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- Similar to Mattila's construction.

# Dot products in higher dimensions

**Theorem (Iosevich, S., 2020)**

*For any $s < \frac{d+1}{2}$, there exists a measure $\mu$ on $[0,2]^d$ with $\dim_{\mathcal{H}} supp(\mu) = s$, that will fail the analog of Falconer's estimate for dot products.*

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- Based on the Valtr construction, but with more arithmetic complexity, and tougher energy estimates.

# Dot product construction

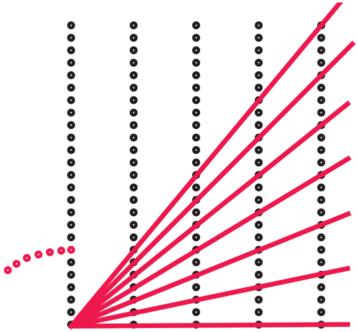

Figure: Similar to the Valtr construction, but now we need to intersect points with lines. Here we have a family of $m$ red lines, of $m$ slopes. These red lines are the set of points that have dot product one with the red points.

# Pinned dot products

$$\Pi_x(E) = \{x \cdot y : y \in E\}.$$

Theorem (Iosevich, Taylor, Uriarte-Tuero, 2016)

*For any* $E \subseteq \mathbb{R}^d$, *with* $\dim E = s > \frac{d+1}{2}$, *the Lebesgue measure of* $\Pi_x(E)$ *is positive.*

# Edge weighted trees

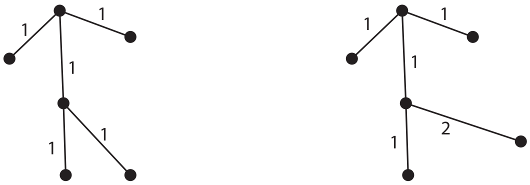


Figure: Trees are acyclic connected graphs. These two trees have the same shape, but different weights.

# Continuous trees - distances

$$\Delta_x(E) = \{|x - y| : y \in E\}.$$

Theorem (Ou and Taylor, 2020)

*Let $E \subseteq R^2$ be a compact set satisfying $\dim_{\mathcal{H}}(E) > \frac{5}{4}$, then there exists a point $x \in E$ such that for all integers $k \geq 2$, we have that any $k$-tree $T$ of any shape pinned at any vertex has a positive $k$-dimensional Lebesgue measure of distinct edge weights determined by distances.*

# Continuous trees - dot products

Corollary (Bright, Marshall, S., 2023+)

*Let* $E \subseteq R^2$ *be a compact set satisfying* $\dim_{\mathcal{H}}(E) > \frac{3}{2}$, *then there exists a point* $x \in E$ *such that for all integers* $k \geq 2$, *we have that any* $k$*-tree* $T$ *of any shape pinned at any vertex has a positive* $k$*-dimensional Lebesgue measure of distinct edge weights determined by dot products.*

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- Simple proof for Ahlfors regular by modifying Moser (1952).

# Dot product trees in higher dimensions

Corollary (Bright, Marshall, S., 2023+)

*For any* $s < \frac{d+1}{2}$*, and tree* $T$ *on* $k$ *edges, there exists a measure* $\mu$ *on* $[0,2]^d$ *with* $\dim_{\mathcal{H}} supp(\mu) = s$*, and a set of edge weights* $\vec{w}$ *so that the analog of Falconer's estimate for* $T$ *with dot product edge weights will fail.*

- In $[0,1]^d$, for $s < \frac{d+1}{2}$, $I_s(\mu) < \infty \not\Rightarrow$

$$\mu^{k+1}\{(x_j, y_j) : w_j \leq x_j \cdot y_j \leq w_j + \epsilon, j = 1, \ldots, k\} \lesssim \epsilon^k.$$

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- Uses the construction in Iosevich and S., (2020).

# Dot product trees in higher dimensions

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- Can be slightly more general than paths.
- Related to work by Barker and S., later improved by Lund.

THANKS! ^_^