Common Fundamental Domains

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The classical Steinhaus question

- Steinhaus (1950s): Are there $A, B \subseteq \mathbb{R}^2$ such that

$$|\tau A \cap B| = 1, \quad \text{for every rigid motion } \tau?$$

Are there two subsets of the plane which, no matter how moved, always intersect at exactly one point?

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- Sierpiński, 1958:

Yes.

The classical Steinhaus question

- Equivalent:

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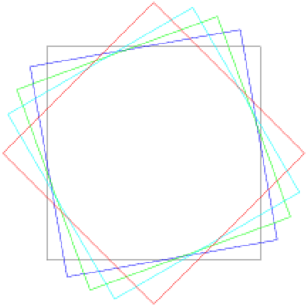
- In tiling language:

$$\rho A \oplus B = \mathbb{R}^2, \quad \text{for all rotations } \rho.$$

Every rotation of A tiles (partitions) the plane when translated at the locations B.

Fixing $B = \mathbb{Z}^2$: the lattice Steinhaus question

- Can we have $\rho A \oplus \mathbb{Z}^2 = \mathbb{R}^2$ for all rotations ρ?



Can a domain behave simultaneously like all those squares?

- Equivalent: A is a fundamental domain of all $\rho\mathbb{Z}^2$.
  Or, A tiles the plane by translations at any $\rho\mathbb{Z}^2$.

The lattice Steinhaus question

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- Results (in the negative direction) by
  Sierpiński (1958), Croft (1982), Beck (1989),
  K. (1996):
  "Best" so far: (K. & Wolff (1999))
  If such a measurable A exists then it must be large at infinity:

$$\int_A |x|^{\frac{46}{27}+\epsilon}\, dx = \infty.$$

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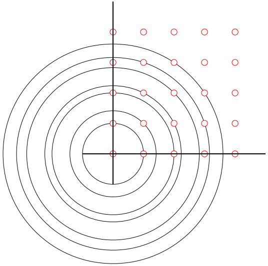
- In higher dimension:
  K. & Wolff (1999), K. & Papadimitrakis (2002):
  $\Longrightarrow$ No measurable Steinhaus sets exist for $\mathbb{Z}^d$, $d \geq 3$.

  No Jackson - Mauldin analogue for $d \geq 3$.

- For A to have the Steinhaus property it is equivalent



that $\widehat{\mathbf{1}_A}$ must vanish on all circles through lattice points.

The zeros of the Fourier Transform

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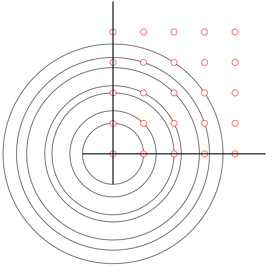

  that $\widehat{\mathbf{1}_A}$ must vanish on all circles through lattice points.

- Too many zeros imply strong decay of $\widehat{\mathbf{1}_A}$ near infinity.

  This implies (uncertainty principle) slow decay of $\mathbf{1}_A$ near infinity.

Lattice Steinhaus for finitely many lattices

- Given lattices $\Lambda_1, \ldots, \Lambda_n \subseteq \mathbb{R}^d$ all of volume 1 can we find measurable A which tiles with all Λ_j?

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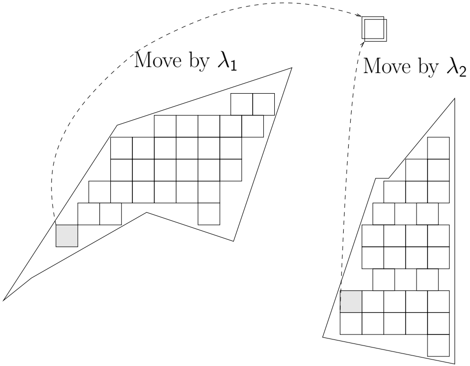
Generically yes!



- If the sum $\Lambda_1^* + \cdots + \Lambda_n^*$ is direct then Kronecker-type density theorems allow us to rearrange a fundamental domain of one lattice to accomodate the others.

Lattice Steinhaus for finitely many lattices

Question

Is there a *bounded* common tile for $\Lambda_1, \ldots, \Lambda_N$?

An application in Gabor analysis

- If K, L are two lattices in $\mathbb{R}^d$ with

$$\operatorname{vol} K \cdot \operatorname{vol} L = 1,$$

can we find $g \in L^2(\mathbb{R}^d)$, such that the (K, L) time-frequency translates

$$g(x-k)e^{2\pi i \ell \cdot x}, \quad (k \in K, \ell \in L)$$

form an orthogonal basis of $L^2(\mathbb{R}^d)$?

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form an orthogonal basis of $L^2(\mathbb{R}^d)$?
- Han and Wang (2000):
Since $\text{vol}\,(L^*) = \text{vol}\,(K)$ let $g = \mathbf{1}_E$ where
E is a **common tile** for K, L^*.
- Then L forms an orthogonal basis for $L^2(E)$.
- Space partitioned in K-copies of E and on each copy L is an orthogonal basis.

Multi-tiling functions

- A function f tiles with the set of translates Λ if

$$\sum_{\lambda \in \Lambda} f(x - \lambda) = \text{const.} \quad \text{a.e. } x \in \mathbb{R}^d.$$

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- We can find a common tiling function f for any set of lattices

$$\Lambda_1,\ldots,\Lambda_N \subseteq \mathbb{R}^d.$$

Just take (the D_j are fundamental domains of Λ_j)

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- For such an f if $\operatorname{vol} \Lambda_j \gtrsim 1$ then

$$\operatorname{diam} \operatorname{supp} f \gtrsim N.$$

Multi-tiling functions: diameter lower bounds

- (K. and Wolff, 1997): If $f \in L^1(\mathbb{R}^d)$, with $\int f \neq 0$, tiles $\mathbb{R}^d$ with $\Lambda_1, \ldots, \Lambda_N$, and

$$\Lambda_i \cap \Lambda_j = \{0\} \text{ and } \mathrm{vol}\, \Lambda_j \sim 1$$

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then
$$\operatorname{diam} \operatorname{supp} f \gtrsim N^{1/d}.$$

Question

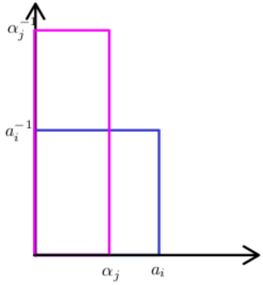
What is the smallest $\operatorname{diam} \operatorname{supp} f$?

We know
$$N^{1/d} \lesssim \operatorname{diam} \operatorname{supp} f \lesssim N.$$
at least when $\Lambda_i \cap \Lambda_j = \{0\}$.

Multi-tiling functions: a case of large diameter

- Take $\alpha_1, \ldots, \alpha_N \in (\frac{1}{2}, 1)$ to be $\mathbb{Q}$-linearly independent and

$$\Lambda_j = \mathbb{Z}(\alpha_j, 0) + \mathbb{Z}(0, \alpha_j^{-1}), \quad \Lambda_j^* = \mathbb{Z}(\alpha_j^{-1}, 0) + \mathbb{Z}(0, \alpha_j).$$



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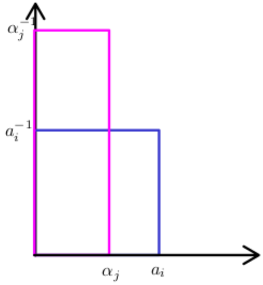

- f tiles with all $\Lambda_j \implies \widehat{f} \equiv 0$ on Λ_j^*.
  $\widehat{f}$ has zeros of density $\gtrsim N$ along the axes. So

$$\mathrm{diam\,supp}\, f \gtrsim N. \quad \text{(K. \& Papageorgiou, 2022)}$$

Question

Is there any case of "generic" lattices with a common tile f s.t.

$$\operatorname{diam} \operatorname{supp} f = o(N)?$$

Multi-tiling functions: the volume of the support

- If $f = \mathbf{1}_{D_1} * \cdots * \mathbf{1}_{D_N}$ or (more generally)

$$f = f_1 * \cdots * f_N, \quad \text{where } f_j \geq 0 \text{ tiles with } \Lambda_j \tag{1}$$

then

$$\operatorname{supp} f = \operatorname{supp} f_1 + \cdots + \operatorname{supp} f_N$$

and (Brunn - Minkowski inequality)

$$|\operatorname{supp} f| \geq \left(|\operatorname{supp} f_1|^{1/d} + \cdots + |\operatorname{supp} f_N|^{1/d}\right)^d \gtrsim N^d.$$

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Question

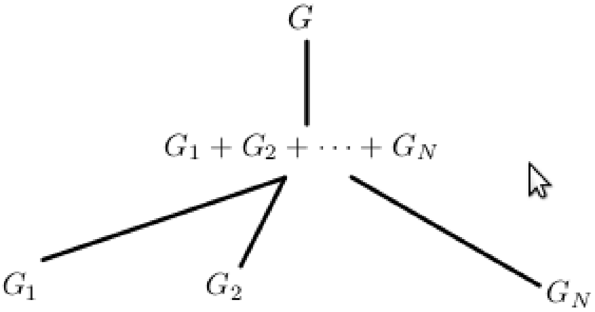
What if we drop nonnegativity from (1)?

What if f is *any* common tile of the Λ_j, not given by (1)?

Multi-tiling sets: Giving up measurability

- If $G_1, \ldots, G_N$ are subgroups of G it is always enough to find a common fundamental domain (a common tile) of the G_j in

$$G_1 + \cdots + G_N.$$



- (K. 1997) If the lattices $\Lambda_1, \ldots, \Lambda_N$ in $\mathbb{R}^d$ have
  (a) *the same volume* and
  (b) a *direct sum*
  then they have a bounded common fundamental domain.

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- (K. 1997) If the lattices $\Lambda_1, \ldots, \Lambda_N$ in $\mathbb{R}^d$ have
  (a) *the same volume* and
  (b) a *direct sum*
  then they have a bounded common fundamental domain.
- A consequence of the "marriage" theorem:

Theorem

If $\operatorname{vol}\Lambda_i = \operatorname{vol}\Lambda_j$ *then there is a bijection* $f_{ij} : \Lambda_i \to \Lambda_j$ *with*

$$|x - f(x)| \text{ bounded.}$$

Equal lattice density necessary for boundedness

- Suppose
$$\Lambda_1 = \mathbb{Z}^d \text{ and } \Lambda_2 = \alpha\mathbb{Z}^d \ (\alpha \text{ irrational}, \ \alpha > 1).$$
  Then Λ_1, Λ_2 have no bounded common fundamental domain.

  No measurability assumed!

Proof for $d = 1$

- If F is a bounded FD in $G = \Lambda_1 + \Lambda_2 = \{m + n\alpha : \ m, n \in \mathbb{Z}\}$:

$$F = m_i - n_i\alpha : i = 1, 2, \ldots \subseteq [-M, M].$$

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- Restricting $-R \leq m \leq R$ we get

$$|m - n_m\alpha| \leq M.$$

or

$$-\frac{R+M}{\alpha} \leq n_m \leq \frac{R+M}{\alpha}.$$

- $\sim 2R$ values of m correspond to only $\sim \frac{2}{\alpha}R$ values of n_m
  Contradiction, as all n_m must be different (K. & Papageorgiou, 2022).

- G_1, G_2 subgroups of G, $f: G \to \mathbb{R}^{\geq 0}$ s.t.

$$\forall x \in G: \quad \sum_{g_1 \in G_1} f(x - g_1) = |G_1|, \quad \sum_{g_2 \in G_2} f(x - g_2) = |G_2|.$$

For example $f(x) \equiv 1$.

Tiling finite abelian groups with a function

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Question

How small can $|\mathrm{supp}\, f|$ be?

- Write

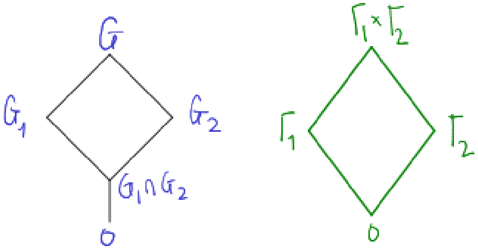
$$S^G_{G_1,G_2} = \min\left\{|\mathrm{supp}\, f| : \ f * \mathbf{1}_{G_1} \equiv |G_1|\mathbf{1}_G, \ \ f * \mathbf{1}_{G_2} \equiv |G_2|\mathbf{1}_G\right\}.$$

- Always $S^G_{G_1,G_2} \geq \max\{[G:G_1],[G:G_2]\}$.

Reduction to product groups

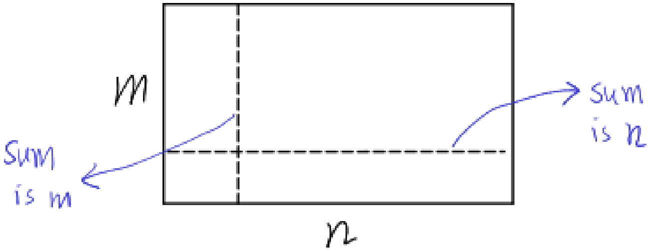
- If $\Gamma = G/(G_1 \cap G_2)$, $\Gamma_i = G_i/(G_1 \cap G_2)$ then

$$S^G_{G_1,G_2} = S^\Gamma_{\Gamma_1,\Gamma_2}. \tag{2}$$



- Can assume: $G = G_1 \times G_2$.

- Group structure irrelevant.



- Find $m \times n$ matrix A with
  row sums equal to n, column sums equal to m.
- Minimize the support. Call $S(m, n)$ the minumum.

The problem in matrix form

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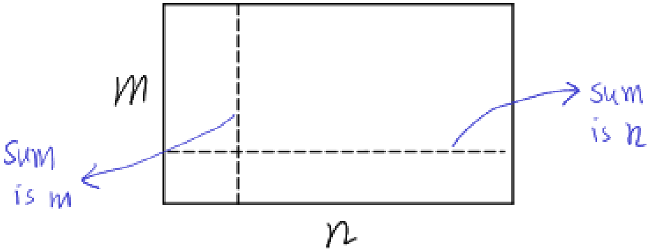


- Find $m \times n$ matrix A with

  row sums equal to n, column sums equal to m.
- Minimize the support. Call $S(m, n)$ the minumum.
- Statisticians call these *copulas* and use them a lot.
  A generalization of doubly stochastic matrices.

The case m divides n

$$m\left\{\underbrace{\begin{bmatrix} \overbrace{m\cdots m}^{k} & \cdots & \cdots \\ \cdots & \overbrace{m\cdots m}^{k} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \overbrace{m\cdots m}^{k} \end{bmatrix}}_{km}\right.$$

- Smallest possible support, since we must have ≥ 1 element/column.

$$S(km, m) = km.$$

The case $n = km + 1$

$$
m\left\{ \vphantom{\begin{bmatrix} 1 \\ 1 \\ \\ 1 \end{bmatrix}} \right.
\underbrace{\begin{bmatrix} 1 & \overbrace{m\cdots m}^{k} & \cdots & \cdots \\ 1 & \cdots & \overbrace{m\cdots m}^{k} & \cdots \\ & \cdots & \cdots & \cdots \\ 1 & \cdots & \cdots & \overbrace{m\cdots m}^{k} \end{bmatrix}}_{km+1}
$$

- Also smallest possible support, since $A_{ij} \leq m$ implies

  at least $k + 1$ terms per row,

  so

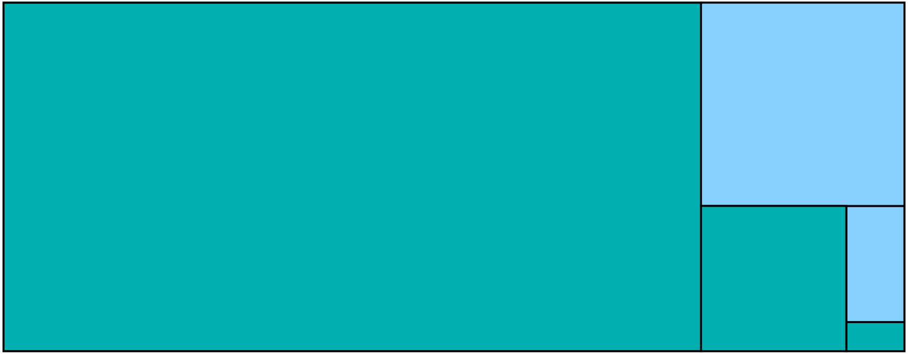
  $$S(km + 1, m) = (k + 1)m = m + (km + 1) - 1.$$

  (K. & Papageorgiou, 2022)

The general case: Loukaki, 2022, Etkind and Lev, 2022

Theorem

$S(m, n) = m + n - \gcd(m, n)$



Tiling $\mathbb{R}$ with two lattices: a lower bound for the length

- Suppose $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$ is measurable and tiles with both $\Lambda_1 = \mathbb{Z}$ and with $\Lambda_2 = \alpha\mathbb{Z}$, where $\alpha \in (0, 1)$:

$$\sum_{n\in\mathbb{Z}} f(x-n) = 1, \quad \sum_{n\in\mathbb{Z}} f(x-n\alpha) = \frac{1}{\alpha}, \quad \text{for almost every } x \in \mathbb{R}. \tag{3}$$

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Then

$$\left|\operatorname{supp} f\right|\ge\left\lceil\frac{1}{\alpha}\right\rceil\alpha\ge 2\alpha. \tag{4}$$

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  (K. & Papageorgiou, 2022)
- When $\alpha = 1 - \epsilon$: convolution $\mathbf{1}_{[0,1]} * \mathbf{1}_{[0,\alpha]}$ is almost optimal.
- When $\alpha = \frac{1}{2} + \epsilon$ there is a big gap $1 + 2\epsilon$ to $3/2 + \epsilon$.

Question

What is the smallest possible length of $\operatorname{supp} f$ which tiles with $\mathbb{Z}$ and $\alpha\mathbb{Z}$?

Tiling $\mathbb{R}$ with two lattices: Etkind and Lev, 2022

$\sum_{k\in\mathbb{Z}} f(x-k\alpha) = p, \quad \sum_{k\in\mathbb{Z}} f(x-k\beta) = q$. What about the measure of $\operatorname{supp} f$?

- $\alpha/\beta \notin \mathbb{Q}$
  - For all $p, q \in \mathbb{C}$ there is measurable f with $|\operatorname{supp} f| \le \alpha+\beta$
  - If $p/q \notin \mathbb{Q}^+$ then for any f must have $|\operatorname{supp} f| \ge \alpha+\beta$.
  - If $f \ge 0$ or $f \in L^1$ or f has bounded support then $p/q = \beta/\alpha$, $|suppf| \ge \alpha+\beta$.
  - If $p/q \in \mathbb{Q}^+$, $\gcd(p,q)=1$ we can have

    $$|\operatorname{supp} f| < \alpha+\beta-\min\left\{\frac{\alpha}{q}, \frac{\beta}{p}\right\}+\epsilon$$

    and must have

    $$|\operatorname{supp} f| > \alpha+\beta-\min\left\{\frac{\alpha}{q}, \frac{\beta}{p}\right\}$$

- $\alpha/\beta \in \mathbb{Q}^+$ and simplifying to $\alpha=n, \beta=m$, with $\gcd(n,m)=1$.

  Then $p/q = m/n$ and the least possible $|\operatorname{supp} f|$ is $n+m-1$.

Subgroups in a finite abelian group: Aivazidis, Loukaki and Sambale, 2023

- If $A_1, \ldots, A_t$ are *complemented* isomorphic subgroups of G and the smallest prime divisor of $|A_1|$ is $\geq t$ then they have a common complement in G.

  $A \subseteq G$ is *complemented* if some FD of A in G is a subgroup of G (called *complement* of A).

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  $A \subseteq G$ is *complemented* if some FD of A in G is a subgroup of G (called *complement* of A).
- If $A, B, C \subseteq G$ are cyclic groups of same order then they have a commond FD in G if and only if the following does not hold:
  $|A| = |B| = |C|$ is even and the product of their 2-Sylow subgroups $A_2B_2C_2$ satisifies
  $$A_2B_2C_2/I = A_2/I \times B_2/I = A_2/I \times C_2/I = B_2/I \times C_2/I$$
  where $I = A_2 \cap B_2 \cap C_2$.

Diameter: lattices with many relations

- **Main observation:** $\Lambda_1, \ldots, \Lambda_N \supseteq \Lambda$ and D is a FD of Λ then

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- Let G be a subgroup of $\mathbb{Z}_p^d$. Define the lattice

$$\Lambda_G = (p\mathbb{Z})^d + G,$$

which contains $\Lambda = (p\mathbb{Z})^2$ with FD

$$[0, p)^d \text{ of diameter } \sqrt{d}p.$$

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- There are
$$\frac{p^d - 1}{p - 1} \sim p^{d-1} =: N$$
different cyclic subgroups G of $\mathbb{Z}_p^d$.

Back to the diameter: an example, continued

- We find $\operatorname{vol}\Lambda_G$ by its density

$$\operatorname{vol}\Lambda_G = \frac{\operatorname{vol}(p\mathbb{Z})^d}{|G|} = \frac{p^d}{p} = p^{d-1} = N.$$

Back to the diameter: an example, continued

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- Shrink everything by $N^{-1/d}$ so that

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- $f(x) := \mathbf{1}_{[0,p)^d}(N^{1/d} x)$ is a common tile for the Λ'_G of diameter

$$\sqrt{d} p \cdot N^{-1/d} = \sqrt{d} N^{\frac{1}{d-1}} N^{-\frac{1}{d}} = \sqrt{d}\ N^{\frac{1}{d(d-1)}}.$$

(K. & Papageorgiou, 2022)

Unconditional lower bounds for the diameter?

Question

Derive a lower bound, growing with N, for

$$\operatorname{diam}\operatorname{supp} f$$

where

$$f \text{ tiles with } \Lambda_1, \ldots, \Lambda_N$$

and $\operatorname{vol}\Lambda_j = 1$.

Diameter: The case $d = 1$.

- Previous construction gives nothing in dimension $d = 1$.

Theorem

We can find N lattices $\Lambda_j \subseteq \mathbb{R}$ of with $\mathrm{vol}\,\Lambda_j \sim 1$ and a function f with $\int f > 0$ and supported in an interval of length

$$\frac{N}{\log^{0.086\cdots} N}$$

which tiles with all Λ_j.

For any $\epsilon > 0$ any such function f must have

$$\mathrm{diam}\,\mathrm{supp}\, f \gtrsim_\epsilon N^{1-\epsilon}.$$

(K. & Papageorgiou, 2022)

Diameter: The case $d = 1$, continued

- Define
$$\Lambda_j = \lambda_j \mathbb{Z} = \frac{1}{N+j}\mathbb{Z}, \quad j = 1, 2, \ldots, N.$$
Then
$$\Lambda_j^* = (N+j)\mathbb{Z},$$
with union $U = \bigcup_{j=1}^{N} (N+j)\mathbb{Z}$.
- f tiles with all $\Lambda_j \iff \widehat{f}$ vanishes on $U \setminus \{0\}$.

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- *Erdős, 1935*: The integers divisible by one of $N+1, N+2, \ldots, 2N$ have density $\to 0$ as $N \to \infty$.
- *Tenenbaum, 1980*: Their density is
$$O\left(\frac{1}{\log^{0.086\cdots} N}\right).$$

Diameter: The case $d = 1$, continued

- So dens $U = O\left(\frac{1}{\log^{0.086\cdots} N}\right)$.

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- *Beurling*: U separated, dens $U < \rho \implies$

$$\exists f : [-\rho, \rho] \to \mathbb{C} \text{ with } \widehat{f} \equiv 0 \text{ on } U, \int f = 1.$$

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- With $\rho = O\left(\frac{1}{\log^{0.086\cdots} N}\right)$ we get a common tile f of support $o(1)$.
- Scale up by a factor of N:

$$f'(x) = f(x/N), \quad \mathrm{diam}\,\mathrm{supp}\, f' = o(N),$$

$$\Lambda_j' = N\Lambda_j = \frac{N}{N+j}\mathbb{Z} \text{ have vol } \sim 1.$$

Diameter: The case $d = 1$: lower bounds

- f tiles with $\Lambda_1, \ldots, \Lambda_N$, $\mathrm{dens}\,\Lambda_j \sim 1$, $\implies$

$$\widehat{f} \text{ vanishes on } \Lambda_1^*, \ldots, \Lambda_N^*.$$

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- *Jensen's formula*: Since $\widehat{f}$ has $\gtrsim N^{2-\epsilon}$ roots in $[-N, N]$ $\implies$

$$\operatorname{diam} \operatorname{supp} f \gtrsim N^{1-\epsilon}.$$

The end

Thank you for your attention!