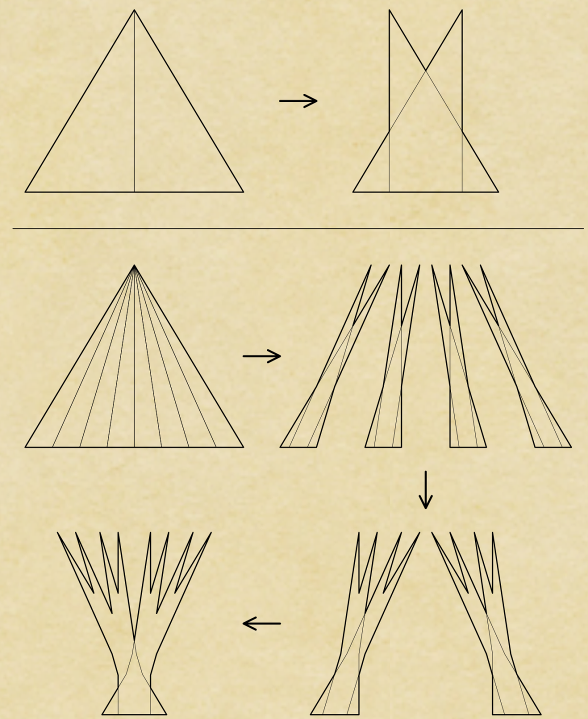
Some structure of Kakeya sets in $\mathbb{R}^3$

Joint with Josh Zahl

A Kakeya set in $\mathbb{R}^d$ is a subset that contains a unit line segment in every direction.



Besicovitch showed (1919) for any $\varepsilon > 0$, there exists a Kakeya set with Lebesgue measure $< \varepsilon$.

Kakeya Conjecture (1917)

Any Kakeya set $K \subseteq \mathbb{R}^d$ has dimension d.

Hausdorff dim

or Minkowski dim

Harmonic analysis

(Applications to PDE, number theory, dynamics, Geometric measure theory)

Kakeya set $\longrightarrow$ Fefferman's Counterexample to Ball multiplier Conjecture

Kakeya Conj $\Longleftarrow$ Stein's restriction Conjecture

$$Ef = \int_{S^{d-1}} e^{2\pi i x \cdot \xi} f(\xi) \, d\sigma(\xi)$$

$$\|Ef\|_{L^p} \le C_p \|f\|_{L^\infty}, \quad \forall p > \frac{2d}{d-1}.$$

History :

$d=2$ Davies 1971

$d \geq 3$ open

Focus on $d=3$

Wolff (1995) : $\dim_H K \geq \frac{5}{2}$.

Katz-Łaba-Tao (2000) : $\exists\, \varepsilon > 0$, $\dim_M K \geq \frac{5}{2} + \varepsilon$

Katz-Zahl (2017) : $\exists\, \varepsilon > 0$, $\dim_H K \geq \frac{5}{2} + \varepsilon$.

W.-Zahl (2024) : Assouad dimension $\dim_A K = 3$

$$\dim_H K \leq \dim_M K \leq \dim_A K .$$

For self similar sets. $\dim_H K = \dim_M K = \dim_A K$.

Assouad dim : " $\exists$ two scales sufficiently separated such that K is large between them".

$\dim_A K := \inf \{ \alpha : \exists\, C, \rho : \forall\, 0 < r < R < \rho ,$
$\sup_{x \in K} |B_R(x) \cap E|_r \leq C \left(\frac{R}{r}\right)^{\alpha} \}$

$|X|_r :=$ minimal number of r-balls required to cover X

δ - thickening $\qquad \delta \to 0$

$\mathbb{T}$: a set of $\delta \times \delta \times 1$ - tubes with one in each δ-separated direction.

$\forall\, \eta > 0$ small. $Y : \mathbb{T} \longrightarrow$ subsets of $\mathbb{R}^3$ is a δ^η- dense shading of $\mathbb{T}$

$T \mapsto Y(T) \subseteq T$

if $\sum_{T \in \mathbb{T}} |Y(T)| \gtrsim \delta^\eta \sum_{T \in \mathbb{T}} |T|$.

δ - discretized Kakeya set : $\boxed{K = \bigcup_{T \in \mathbb{T}} T}$.

Can make K uniform:

$\forall$ dyadic cube Q_r of side length $r \in \delta^{\eta \mathbb{N}}$,

if $Q_r \cap K \neq \phi$ then

$|Q_r \cap K| \approx \frac{|K|}{|K|_r}$.

$\dim_H K = 3$: $\quad \forall\, \varepsilon > 0, \quad |K| \gtrsim_\varepsilon \delta^\varepsilon$.

$\dim_M K = 3$: $\quad \forall\, \varepsilon > 0, \quad \exists\, \varepsilon_1 > 0, \quad \exists\, \rho \in (\delta, \delta^{\varepsilon_1}), \quad |N_\rho K| \gtrsim_\varepsilon \rho^\varepsilon$

$\dim_A K = 3$ $\quad \forall\, \varepsilon > 0, \quad \exists\, \varepsilon_1 > 0, \quad \exists\, \delta < \rho_1 < \delta^{\varepsilon_1} \rho_2 \leq \delta^{\varepsilon_1}$,

$$|N_{\rho_1} K| \gtrsim_\varepsilon \left(\frac{\rho_1}{\rho_2}\right)^\varepsilon |N_{\rho_2} K|.$$

Why Wolff's estimate is hard to improve ?

Wolff Axiom for $\mathbb{T}$: $|\mathbb{T}| \simeq \delta^{-2}$, any $\delta \times \rho \times 1$ box contains $\leq \frac{\rho}{\delta}$ tubes of $\mathbb{T}$.

Wolff's estimate holds for any set of δ-tubes satisfying Wolff Axiom.

(not necessarily in $\mathbb{R}^3$!)

Kakeya set satisfies Wolff Axiom.

Key Obstacle: Heisenberg group

$$H = \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : \operatorname{Im}(z_1) = \operatorname{Im}(z_2 \bar{z}_3) \}$$

$\forall s, t \in \mathbb{R}, \alpha \in \mathbb{C}$, the line $\ell_{s,t,\alpha} := \{ (\bar{\alpha} z + t,\ z,\ sz + \alpha) : z \in \mathbb{C} \} \subseteq H$.

$$\mathbb{T} = \{ N_\delta \ell_{s,t,\alpha} \}_{\substack{s,t \in \mathbb{R} \\ \alpha \in \mathbb{C}}}$$ satisfies Wolff Axiom.

But $\dim_{\mathbb{R}} H = 5 = \frac{5}{2} \dim_{\mathbb{R}} \mathbb{C}$.

To overcome this obstacle:

ⓐ use $\mathbb{T}$ contains a δ-tube in every direction (Katz-Łaba-Tao, Dvir)

→ Difficult to induct. does not preserve under thickening or zooming in.

ⓑ use tubes are in $\mathbb{R}^3$, not in $\mathbb{C}^3$ (Katz-Zahl, W.-Zahl)

Key ingredient for ⓑ

Bourgain's discretized sum·product theorem (Confirming a conj of Katz-Tao)

$0 < s < 1$, $\exists\, \varepsilon > 0$, for any $A \subseteq [1,2]$ satisfying

$$|A \cap B_r|_\delta \lessapprox r^s |A|_\delta, \quad \forall\, r \in [\delta, 1].$$

we have

$$\max\{|A+A|_\delta,\ |A \cdot A|_\delta\} \gtrsim |A|_\delta^{1+\varepsilon}.$$

$|A|_\delta$ = min # of δ-balls requires to cover A.

i.e. $\mathbb{R}$ does not contain a subring of Hausdorff dim $s \in (0,1)$.

(Miller - Edgar)

Set up:

$\mathbb{T}$ satisfies Convex Wolff Axiom if $\forall$ any convex set U,

$$\mathbb{T}[U] := \{ T \in \mathbb{T} : T \subseteq U \}$$

$$\# \mathbb{T}[U] \lesssim |U| \cdot \# \mathbb{T} \qquad (\Rightarrow \# \mathbb{T} \gtrsim \delta^{-2})$$

Conj: For any set $\mathbb{T}$ of distinct δ-tubes in $\mathbb{R}^3$ satisfying Convex Wolff Axiom

$$\Big| \bigcup_{T \in \mathbb{T}} T \Big| \gtrsim C_\varepsilon \, \delta^{\varepsilon} .$$

Remarks:

- This conjecture, if true, is an "if and only if" condition. More general than Kakeya conjecture.
- In $\mathbb{R}^4$, not true because of $\{xy - zw = 1\}$.

  + polynomial convex Wolff Axiom ?

$\mathbb{T}$ satisfies Convex Wolff Axiom if $\forall$ any convex set U,

$$\mathbb{T}[U] := \{ T \in \mathbb{T} : T \subseteq U \}$$

$$\# \mathbb{T}[U] \lesssim |U| \cdot \# \mathbb{T} \qquad (\Rightarrow \# \mathbb{T} \gtrsim \delta^{-2})$$

<u>Thm</u> (W. - Zahl 2024+)

$\forall \, \varepsilon > 0$, $\exists \, \varepsilon_1 > 0$ such that the following holds for $\delta > 0$ sufficiently small.

For any set $\mathbb{T}$ of distinct δ-tubes satisfying Convex Wolff Axiom,

$\exists \; \delta \leq \rho_1 < \rho_2 \delta^{\varepsilon_1} \leq \delta^{\varepsilon_1}$. $K = \bigcup_{T \in \mathbb{T}} T$,

$$|N_{\rho_1} K| \gtrsim \left(\frac{\rho_1}{\rho_2}\right)^{\varepsilon} |N_{\rho_2} K| .$$

Digest the notation :

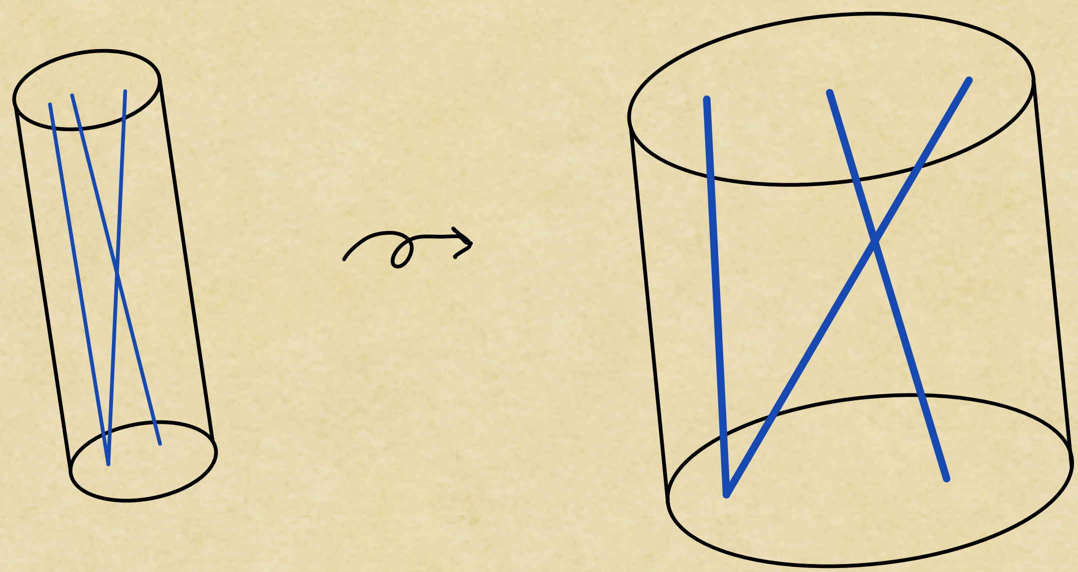
<u>Set up:</u>

$\mathbb{T}$ satisfies Convex Wolff Axiom if $\forall$ any convex set U,

$$\mathbb{T}[U] := \{ T \in \mathbb{T} : T \subseteq U \}$$

$$\# \mathbb{T}[U] \lesssim |U| \cdot \# \mathbb{T} \qquad (\Rightarrow \# \mathbb{T} \gtrsim \delta^{-2})$$

- Convex Wolff Axiom preserves under thickening
- ~~Zooming~~ in ?



<u>Dichotomy</u> : find a "worst" $\mathbb{T}$.

- Either $\mathbb{T}$ satisfies Convex Wolff Axiom at every scales : $\forall\ \rho \in (\delta, 1)$, $\forall\ T_\rho \in \mathbb{T}_\rho$ (ρ-tubes covering $\mathbb{T}$)

  $\mathbb{T}[T_\rho]$ satisfies Convex Wolff Axiom.

  In this case, apply an earlier result (W.-Zahl) on Sticky Kakeya sets to show $\dim_H K = 3$. $K = \bigcup_{T \in \mathbb{T}} T$

  (This is where Bourgain's discretized sum product is used.)

  (Orponen-Shmerkin-W.)

- or K has Assouad dimension 3.

To prove the dichotomy,

Let $\mathbb{T}$ be a set of δ-tubes satisfying Convex Wolff Axiom with $K = \bigcup_{T \in \mathbb{T}} T$ having smallest Assouad dimension, among these minimizers choose one with $|\mathbb{T}| = \delta^{-\alpha}$. α largest.

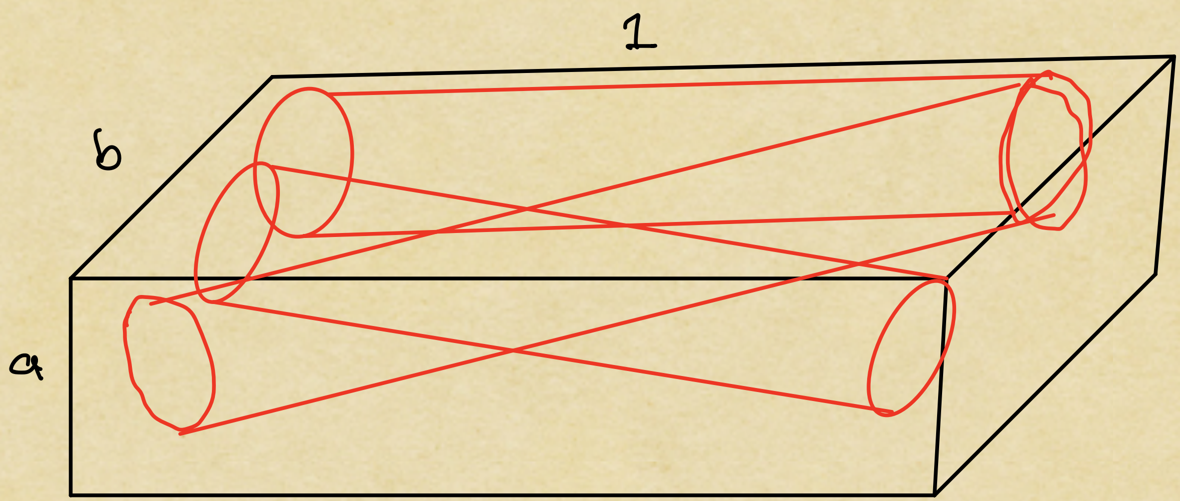
$$\Rightarrow \quad |\mathbb{T}_\rho| \lessapprox \rho^{-\alpha}, \quad |\mathbb{T}[T_\rho]| \lessapprox \left(\frac{\delta}{\rho}\right)^{-\alpha} \quad \forall\, \rho \in (\delta, 1) \quad \text{———} \; (*)$$

$\phi_{T_\rho} : T_\rho \to [0,1]^3$. If $\phi_{T_\rho}(\mathbb{T}[T_\rho])$ fails Convex Wolff Axiom,

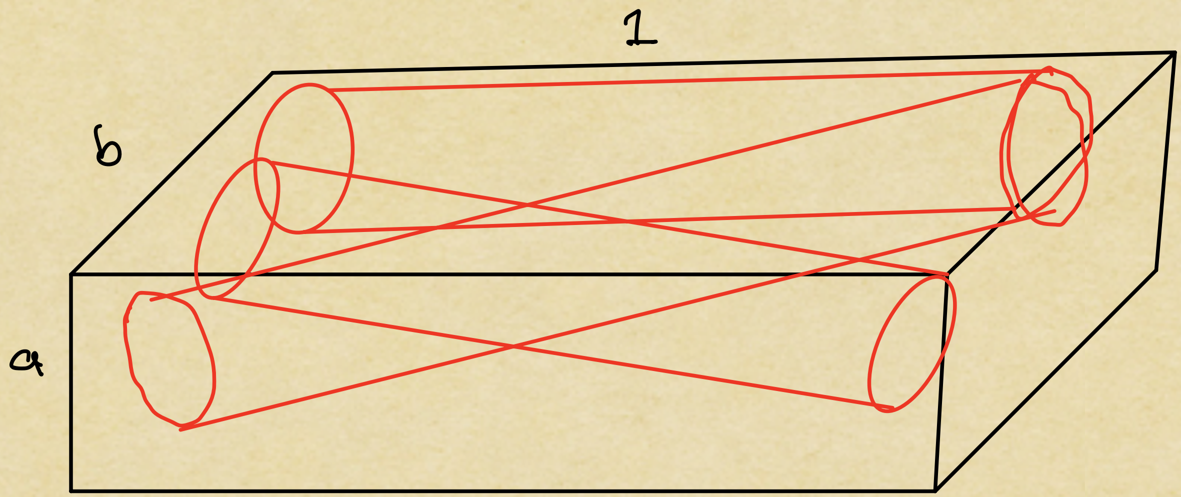
Then there exist (many) $a \times b \times 1$ boxes W, $a \ll b$, such that

$$|\mathbb{T}[W]| \text{ is large.}$$

$(*) \Rightarrow W$ contains many T_a : $|N_a K \cap W| \approx |W|$



W contains many T_a : $|N_a K \cap W| \approx |W|$



- If $\{W\}$ intersect transversally, take $\rho_1 = a$, $\rho_2 = a\delta^{-\varepsilon_1}$

- Otherwise $\{W\}$ intersect tangentially, replace W with a larger $\hat{a} \times \hat{b} \times 1$-box : $\frac{\hat{a}}{\hat{b}} = \frac{a}{b}$.

Iterate until $\tilde{b} \simeq 1$ and estimate $N_{\hat{a}} K = \cup \tilde{W}$ directly.

Some thoughts on the set up.

Restricted projection problem (introduced by Fässler - Orponen)

Given a smooth curve $V(t) \subseteq G(n,m) = \{ m\text{-dim subspace in } \mathbb{R}^n \}$

$P_t : \mathbb{R}^n \longrightarrow V(t)$. orthogonal projection.

$E \subseteq \mathbb{R}^n$. Borel set. What is $\sup_{t \in [0,1]} \dim_H P_t E$?

Ex 1 :

$$\gamma_1(t) = (1, 0, t, 0), \quad \gamma_2(t) = (0, 1, 0, t)$$

$$V(t) = \mathrm{Span}\langle \gamma_1(t), \gamma_2(t) \rangle \subseteq G(4,2).$$

$$P_t : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$$

$$x \mapsto (x \cdot \gamma_1(t), x \cdot \gamma_2(t))$$

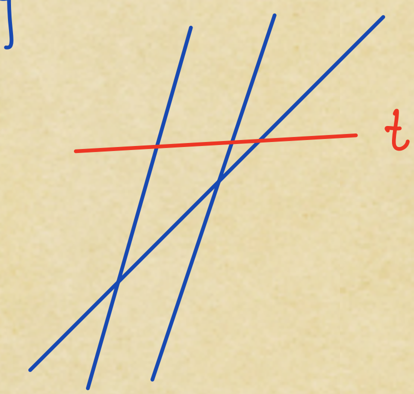
$\exists E$, $\dim_H E = 2$. $P_t E$ is a line $\forall t \in [0,1]$.

Need to add the right assumption $\approx$ Kakeya problem in $\mathbb{R}^3$.

$E = \{(a, b, c, d)\}$: set of parameters for tubes

$P_t E = \{(a+ct, \; b+dt)\}$

= a slice of union of tubes at height t.



Ex 2 : $\gamma_1(t) = (1, t, \frac{t^2}{2!}, \frac{t^3}{3!})$

$\gamma_2(t) = (0, 1, t, \frac{t^2}{2!})$

$$\sup_{t \in [0,1]} \dim_H P_t E = \min\{\dim_H E, 2\}$$

(Gan-Guo-W.)

proof uses harmonic analysis (Bourgain-Demeter-Guth decoupling),

∃? proof without Fourier analysis ?

General Problem

$V(t) \subseteq G(n,m)$, $P_t : \mathbb{R}^n \longrightarrow V(t)$ orthogonal projection.

$E \subseteq \mathbb{R}^n$ Borel set.

① For what $V(t)$ do we have $\sup_{t\in[0,1]} \dim_H P_t E = \min\{\dim_H E, m\}$?

② When ① is false, What reasonable assumptions to add on E s.t. we have a good estimate $\sup_{t\in[0,1]} \dim_H P_t E \geq$?

Thank you !