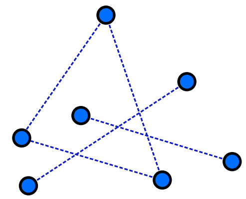
Colorful Intersections & Tverberg Partitions



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Dohyeon Lee
Andreas F Holmsen — KAIST & IBS DIMAG

The Colorful Helly Theorem

Lovász (1970's)
Bárány (1982)

Thm. Let $F_1, \dots, F_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$ such that $C_1 \cap \dots \cap C_{d+1} \neq \emptyset$ for every choice $C_i \in F_i$. Then there is a POINT that intersects every member of one of the F_i.

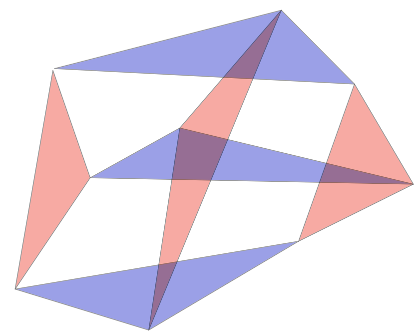
"colorful intersection property"

MONTEJANO'S THEOREM

Montejano (2013)
Montejano - Karasev (2011)
Strausz (2022)

Thm. Consider three red and three blue convex sets in $\mathbb{R}^3$ with the colorful intersection property. Then there is a LINE that intersects every red set or every blue set

one of the colors has a "line transversal"



TVERBERG'S THEOREM

Tverberg (1966)

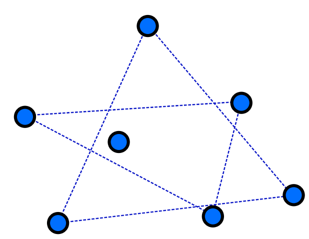
Thm. Let F be a set of $n > (d+1)(r-1)$ points in $\mathbb{R}^d$.

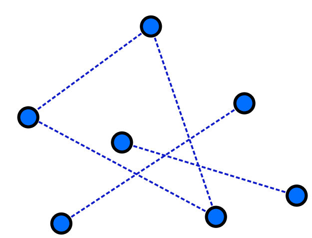
There is a partition $F = A_1 \cup \cdots \cup A_r$ such that

$$\operatorname{conv} A_1 \cap \cdots \cap \operatorname{conv} A_r \neq \emptyset$$

"Tverberg r-partition"

E.g. $d=2$, $r=3$





Our Result

Dobbins – H. – Lee (2024)

Thm. Let $F_1, \dots, F_m$ be families of convex sets in $\mathbb{R}^d$, each of size n, with the colorful intersection property. If $n > (\frac{d}{m}+1)(r-1)$, where r is a prime power, then one of the F_i has a Tverberg r-partition.

Cor. One of the F_i has an $(n-r)$-flat transversal

E.g. $d=3$, $m=2$, $r=2$, $n=3$ Montejano's theorem

$d=10$, $m=7$, $r=3$, $n=5$ $\Rightarrow$ 2-flat transversal

QUESTION

Special case of conjecture
of Martínez - Roldán - Rubin (2018)

Consider 1000 red and 1000 blue convex sets in $\mathbb{R}^3$
with the colorful intersection property.

Is there always a LINE that intersects
4 of the red sets or 4 of the blue sets ?

CONFIGURATION SPACE / TEST MAP SCHEME

"test map"

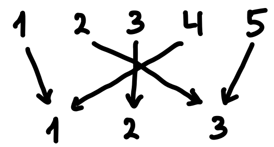
$$\text{"Configuration space"} \xrightarrow{\;f\;} \text{"Test space"}$$

Thm (Volovikov 1996). Let $G = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ with p prime. Let X and Y be fixed point-free G-spaces, where X is n-connected and $Y \simeq S^n$ is finite dimensional. Then there exists no G-equivariant map $X \longrightarrow Y$.

CONFIGURATION SPACE

$V_{n,r}$ set of surjective functions $[n] \to [r]$

E.g.



$\longleftrightarrow \left(\{1,4\}, \{3\}, \{2,5\} \right)$

ordered r-partitions of $[n]$

S_r acts on $V_{n,r}$ by permuting parts

$K_{n,r}$ simplicial complex on $V_{n,r}$:

$\{\varphi_1, \dots, \varphi_j\} \in K_{n,r} \longleftrightarrow$ component-wise nonempty intersections

$$K = K_{n,r}^{*m} = K_{n,r} * \cdots * K_{n,r}$$

Encodes the ordered r-partitions of the F_i

$F_1 \quad \cdots \quad F_m$

SARKARIA'S TENSOR TRICK

Sarkaria (1992)
Bárány-Onn (1997)

$v_i = e_i - \frac{1}{r}\mathbb{1} \in \mathbb{R}^r, \quad v_1 + \cdots + v_r = 0$

matrices w/row sums = 0

$w \in \mathbb{R}^d \longrightarrow L_i(w) = \binom{w}{1} \otimes v_i \in Y \subset \mathbb{R}^{(d+1)\times r}$

S_r acts on Y by permuting columns

For convex set $C \in \mathbb{R}^d$ $\quad L_i C = \{L_i(w) : w \in C\}$

OBS. $F = \{C_1, \dots, C_n\}$ convex sets in $\mathbb{R}^d$

$\varphi \in V_{n,r}$ is a Tverberg r-partition of F

$$\Updownarrow$$

$$0 \in \operatorname{conv}\left(\{L_1 C_i\}_{i\in\varphi^{-1}(1)} \cup \cdots \cup \{L_r C_i\}_{i\in\varphi^{-1}(r)}\right)$$

TEST MAP

vertex $\varphi \in K \longleftrightarrow$ r-partition of some F_i

$\downarrow$ assuming not a Tverberg partition

$$\{L_1 C_i\}_{i\in\varphi^{-1}(1)} \cup \cdots \cup \{L_r C_i\}_{i\in\varphi^{-1}(r)} \subset \{y \in Y : a_\varphi \cdot y > 0\}$$

Define $f : K \longrightarrow Y$ by affine extension

f commutes with the group action

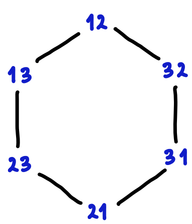
colorful intersections $\Rightarrow (\mathrm{Im}\, f) \cap (e_{d+1} \otimes \mathbb{R}^r) = \emptyset$

projection & normalization $\Rightarrow f : K \longrightarrow S^{d(r-1)-1}$

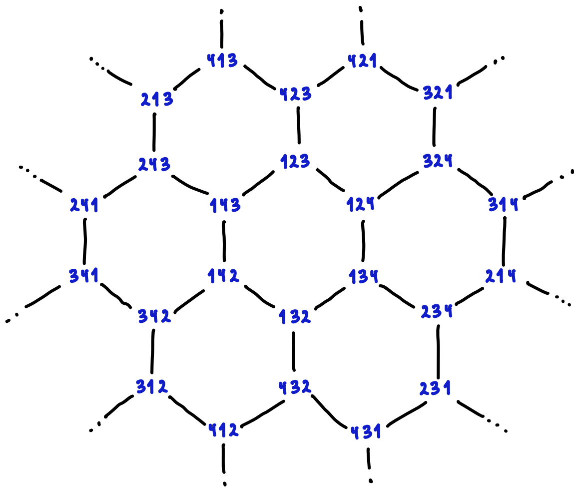
G-equivariant

CONNECTEDNESS OF K

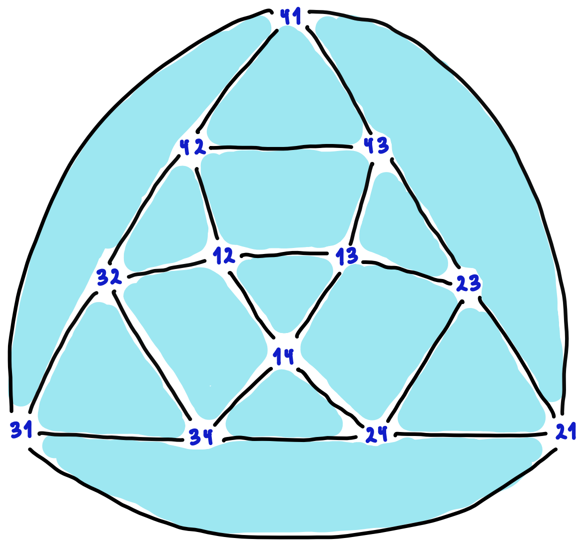
$C_{n,r}$ cell complex of ordered r-partitions of subsets of [n]



$C_{3,2}$



$C_{4,3}$



$C_{4,2}$

$C_{n,r}$ is an $(n-r)$-dimensional regular cell complex

$K_{n,r} \simeq C_{n,r}$ by Quillen's fiber theorem

CONNECTEDNESS OF K

$$K_{n,r} \simeq C_{n,r} \simeq \bigvee_{i\in I} S^{n-r} \Rightarrow (n-r-1)\text{-connected}$$

Quillen's fiber thm

Discrete Morse th.

$$\Rightarrow K = K_{n,r}^{*m} \text{ is } [m(n-r+1)-2]\text{-connected}$$

$> d(r-1)-2$ since $n > \left(\frac{d}{m}+1\right)(r-1)$

contradicts the equivariant map $f: K \to S^{d(r-1)-1}$ $\square$

A CONJECTURE

Our result is GEOMETRIC,

but the proof is TOPOLOGICAL

requires r is prime power

We conjecture the result holds for any integer $r \geq 2$.

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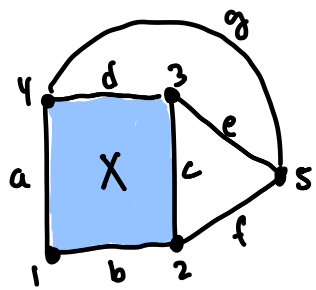
THANK YOU!



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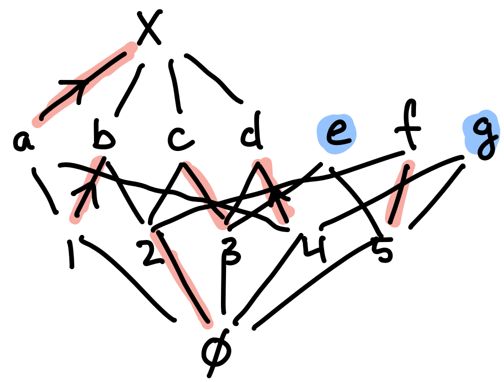

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Wednesday, July 17, 2024

9:30—10:30	Pertti Mattila	Hausdorff dimension of plane sections and general intersections
10:30—11:00		Coffee break
11:00—11:30	Wei-Hsuan Yu	On the size of maximal binary codes with 2, 3, and 4 distances
11:30—12:00	Hai Long Dao	The combinatorics of syzygies
12:00—14:00		Lunch
14:00—19:00		Excursion
19:00		Banquet

Thursday, July 18, 2024

9:30—10:30	Hong Wang	Invited lecture
10:30—11:00		Coffee break
11:00—11:30	Alan Chang	Dividing a set in half
11:30—12:00	Terry Harris	Subsets of vertical planes in the first Heisenberg group
12:00—14:00		Lunch
14:00—14:30	Charlotte Aten	TBA
14:30—15:00	Semin Yoo	Improved upper bounds for the largest size of Diophantine m-tuples
15:00—15:30		Coffee break
15:30—17:30		Small group collaboration

Friday, July 19, 2024

9:30—10:30	Cosmin Pohoata	TBA
10:30—11:00		Coffee break
11:00—11:30	Andreas Holmsen	Colorful intersections and Tverberg partitions
11:30—12:00	Jinha Kim	Star clusters in independence complexes of hypergraphs
12:00—14:00		Lunch
14:45—15:15	Olivine Silier	TBA
15:15—15:45	Matthew Kroeker	The Average Number of Points in a Spanned Plane
15:45—16:15		Coffee break
16:15—17:30		Small group collaboration