Covers of Triangular Grids

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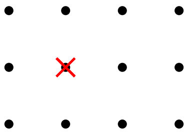
Joint work with Abdul Basit and Paul Horn

Almost Covers of Rectangular Grids

Theorem (Alon–Füredi, 1993)

For sets $S_1, S_2, \cdots, S_n \subset \mathbb{R}$, the minimum number of affine hyperplanes in $\mathbb{R}^n$ needed to cover all but one point of $S_1 \times S_2 \times \cdots \times S_n$ and leave the last point uncovered is

$$\sum_{i=1}^{n}(|S_i| - 1).$$



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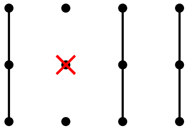

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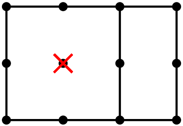
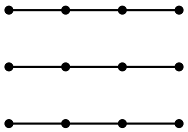


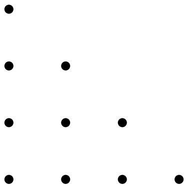
Why remove a point?

If we instead insist on covering every point of $S_1 \times S_2 \times \dots S_n$, then this is a very boring question.



Every point lies on a hyperplane of maximum size!

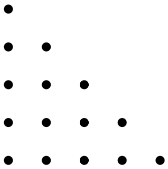
Triangular Grids



Not every point lies on a hyperplane of maximum size!

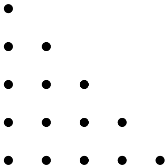
Notation

Let $T_d(n) := \{(x_1, \cdots, x_d) \in \mathbb{Z}_{\geq 0}^d \mid x_1 + \cdots + x_d \leq n - 1\}$.



Let $f(n, d, k)$ denote the minimum number of hyperplanes needed to cover every point of $T_d(n)$ at least k times.

Integer Covering



Theorem (Basit–C.–Horn, 2023+)

For all $n \geq 2$,

$$f(n,2,k) = \begin{cases} n & \text{if } k = 1, \\ 3n/2 & \text{if } k = 2, \\ 9n/4 & \text{if } k = 3, \\ 3n & \text{if } k = 4. \end{cases}$$

Proof for $k = 4$: Upper Bound

Theorem (Basit–C.–Horn, 2023+)

For all $n \geq 2$, $f(n, 2, 4) = 3n$.

Proof.

Our construction consists solely of lines parallel to the sides of the outer triangle.

- Lines $x = i, y = i$, and $x + y = n - 1 - i$ for $i \in \{0, \ldots, \frac{n-1}{3}\}$ have multiplicity 2.
- Lines $x = i, y = i$, and $x + y = n - 1 - i$ for $i \in \{\frac{n-1}{3} + 1, \ldots, \frac{2n}{3} - 1\}$ have multiplicity 1.

$\square$

Proof for $k = 4$: Lower Bound

Theorem (Basit–C.–Horn, 2023+)

For all $n \geq 2$, $f(n,2,4) = 3n$.

Proof.

We proceed by induction to show $f(n,2,4) \geq 3n$.

If we have to use one of the outer lines ($x = 0$, $y = 0$, or $x + y = n - 1$) at least three times, then that means we require at least $f(n-1,2,4) + 3 = (3n-3) + 3$ lines.

Proof for $k = 4$: Lower Bound

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If we use each outer line at most twice, this leaves $3(n-2)$ points on the boundary that need to be covered an additional two times each. Only two of these can be covered at a time by any other line so that forces at least $\frac{3(n-2)(2)}{2} = 3n - 6$ more lines for a total of $3n - 6 + 6 = 3n$. □

Integer Program

$f(d, n, k)$ is the minimum number of hyperplanes needed to cover every point of $T_d(n)$ at least k times each.

This can be interpreted as the optimum of an integer program:

- Variables correspond to how many times each hyperplane is used.
- Constraints correspond to each of the grid points being covered at least k times.

Linear Relaxation

We define $f^*(n, d, k)$ to be the optimum of the linear relaxation. We write $f^*(n, d) := f^*(n, d, 1)$.

$f(n, d, k) \geq f^*(n, d, k) = kf^*(n, d)$.

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Theorem (Basit–C.–Horn, 2023+)

For all integers $j \geq 0$,

$$\begin{cases} f^*(3j+1, 2) = 2j+1, \\ f^*(3j+2, 2) = 2j+1+\dfrac{2j+1}{3j+2}, \\ f^*(3j+3, 2) = 2j+2+\dfrac{j+1}{3j+4}. \end{cases}$$

$$1, \frac{3}{2}, \frac{9}{4}, 3, \frac{18}{5}, \frac{30}{7}, 5, \ldots$$

Fractional Covering: Upper Bound

Theorem (Basit–C.–Horn, 2023+)

$f^*(3j+1,2) = 2j+1$ *for all integers* $j \geq 0$.

$T_2(3j+1) = \{(x,y) \mid x,y \geq 0, x+y \leq 3j\}$. We can cover all these points with the following lines:

- $x = i$ for $i = 0, \cdots, 2j-1$ with weight $\frac{2j-i}{3j}$,
- $y = i$ from $i = 0, \cdots, 2j-1$ with weight $\frac{2j-i}{3j}$, and
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If $i_1, i_2 \leq 2j-1$, (i_1, i_2) is covered with weight $\frac{2j-i_1}{3j}$ by a vertical line and weight $\frac{2j-i_2}{3j}$ by a horizontal line for a total weight of $\frac{4j-i_1-i_2}{3j}$.

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If this is not at least 1, $i_1 + i_2 \geq j+1$ and the point is covered by a diagonal line with weight $\frac{i_1+i_2-j}{3j}$ for a total weight of 1.

Fractional Covering: Lower Bound

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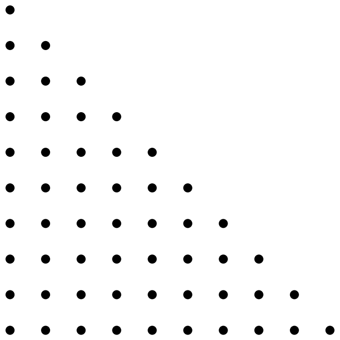

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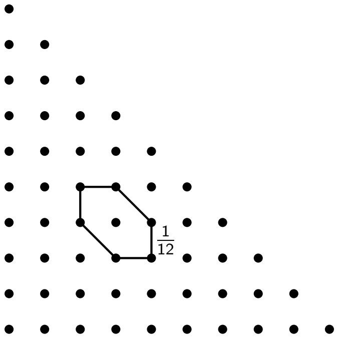

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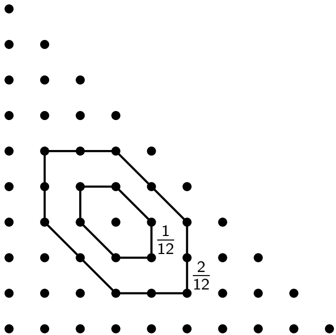

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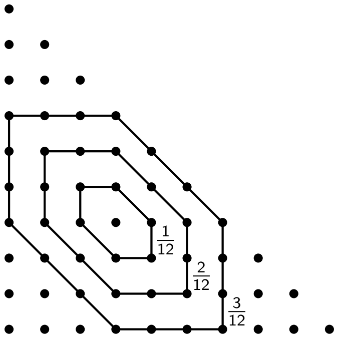

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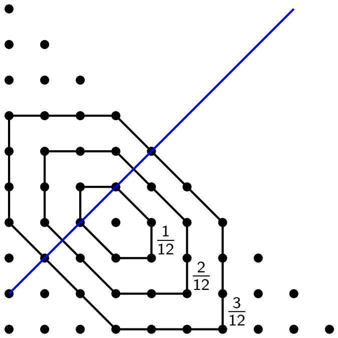

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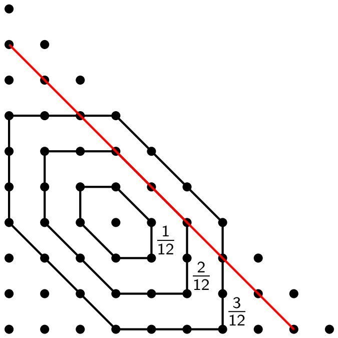

Integer Covering Revisited

We automatically get the bound $f(n, 2, k) \geq kf^*(n, 2)$ but it is not tight.

For example, $f^*(n, 2) = 2n/3 + O(1)$, but $f(n, 2, 4) = 3n$ rather than $8n/3 + O(1)$.

Computations suggest $f(n, 2, k) = C_k n + O(1)$ for some constant C_k and in particular that $C_5 = 18/5$, $C_6 = 30/7$, and $C_7 = 5$.

Conjecture

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For $k \geq 1$,

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- We can translate the upper bound construction for the fractional problem to the necessary upper bound construction for the integer program.
- The desired lower bound on $f(n, 2, k)$ holds under certain natural constraints.

Results

Recall that $f(n,d,k)$ is the minimum number of hyperplanes needed to cover every point of $T_d(n) := \{(x_1, \cdots, x_d) \in \mathbb{Z}^d_{\geq 0} \mid x_1 + \cdots + x_d \leq n-1\}$ at least k times.

Theorem

a) *If* $k \geq 2$ *and* $d \geq 2k-3$, *then*

$$f(n,d,k) = \left(1 + \frac{k-1}{d-k+2}\right) n + O_{d,k}(1),$$

b) *If* $k \geq 3$ *and* $2k-3 \geq d \geq k-2$, *then*

$$f(n,d,k) = \left(2 + \frac{2k-3-d}{2d+3-k}\right) n + O_{d,k}(1).$$

Key Observations

Fix d and k.

1) Suppose you want to show a lower bound of $f(n, d, k) \geq Cn + C'$ via induction on n. It suffices to assume that all *bounding hyperplanes* ($x_i = 0$ or $x_1 + \cdots + x_d = n - 1$) are used fewer than C times.

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2) The intersection of a bounding hyperplane H with $T_d(n)$ is a copy of $T_{d-1}(n)$. Any hyperplane not parallel to H intersects this in an affine subspace of dimension $d - 2$. Thus, the number of hyperplanes needed to cover k times this copy of $T_{d-1}(n)$ without using H is at least $f(n, d - 1, k)$.

Proof Example

We induct on k. Suppose we wish to show that $f(n, 6, 4) = 7n/4 + O(1)$ and we already know that $f(n, 5, 3) = 3n/2 + O(1)$.

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By Observation 1), it suffices to assume that every bounding hyperplane of $T_6(n)$ has multiplicity at most 1. Then excluding the bounding hyperplanes used, each face of the grid, which is a copy of $T_5(n)$, includes an interior copy of $T_5(n - 6)$ whose points have been covered at most once.

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We cannot use anymore bounding hyperplanes so by Observation 2), each of these copies requires at least $f(n-6,5,3) = 3n/2 + O(1)$ hyperplanes to be covered an additional three times. However, no hyperplane will intersect all seven copies of $T_5(n-6)$ that need to be covered, so this requires at least

$$\left(\frac{7}{6}\right)(3n/2 + O(1)) = 7n/4 + O(1).$$

Open Problems

- Is $f(n,3,k) = \left(\frac{k+1}{2}\right) n + O_k(1)$ for odd k and $\left(\frac{k+1}{2} - \frac{1}{2(k+1)}\right) n + O_k(1)$ for even k?
- Determine the asymptotic formula (in terms of n) for general $f(n,d,k)$.

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- Determine the asymptotic formula (in terms of n) for general $f(n,d,k)$.
- Is $f(n,d,k) \geq f^*(k,d)n$ for all n,d,k?
- Does $f(n,d,k) = f(k,d,n)$ for all n,d,k?

Thank you!

Any Questions?