Distance problems and geometric averaging operators

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July 16, 2024

IBS-DIMAG Workshop on Combinatorics and Geometric Measure Theory

The distance set

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- Idea: $E \subseteq \mathbb{R}^d$ large $\implies D(E)$ large (& structured)

The Falconer distance problem

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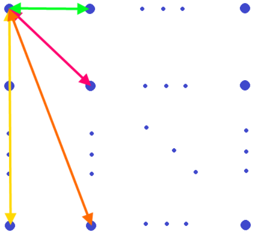
- Falconer's conjecture $\dim_{\mathcal{H}}(E) > \frac{d}{2}$



Encode dimension with measures

- For a compact set $E \subset \mathbb{R}^d$ and $0 < s < \dim_{\mathcal{H}}(E)$ there is a probability measure μ supported on E with

$$\mu(B_r) \lesssim r^s$$

for any ball B_r of radius r. Call μ a *Frostman measure*.

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- Taking s arbitrarily smaller

$$I_s(\mu) = \iint |x-y|^{-s} d\mu(x) d\mu(y) = c_{s,d} \int |\widehat{\mu}(\xi)|^2 |\xi|^{s-d} d\xi < \infty$$

Call $I_s(\mu)$ the *energy integral* of μ.

Distance measure

- Define the distance measure $\delta(\mu)$, supported on $D(E)$, by the relation

$$\int f(r)\, d\delta(\mu)(r) = \iint f\left(|x_1 - x_2|\right) d\mu(x_1) d\mu(x_2)$$

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- Approximate μ by a smooth function μ_ϵ and get

$$\begin{aligned} \int f(r)\, d\delta(\mu_\epsilon)(r) &= \iint f\left(|x_1 - x_2|\right) \mu_\epsilon(x_1)\mu_\epsilon(x_2) dx_1 dx_2 \\ &= \int f(r) \left(\int (\sigma_r * \mu_\epsilon)(x)\mu_\epsilon(x) dx \right) dr \end{aligned}$$

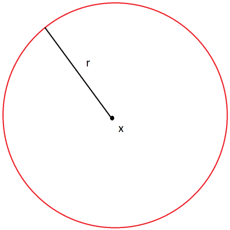
Spherical averaging operator appears

- Distance measure has density given by

$$\delta(\mu_\epsilon)(r) = \int (\sigma_r * \mu_\epsilon)(x)\mu_\epsilon(x)dx = r^{d-1} \langle A_r(\mu_\epsilon), \mu_\epsilon \rangle$$

where A_r is the spherical averaging operator

$$A_r(f)(x) = \frac{1}{r^{d-1}}(\sigma_r * f)(x) = \int_{\mathbb{S}^{d-1}} f(x - ry)d\sigma(y)$$



Many bounds for the spherical averaging operator

- $A_r : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, $p \geq 1$, for example

$$\|A_r(f)\|_{L^1} \leq \int_{\mathbb{S}^{d-1}} \|f\|_{L^1} \, d\sigma(y) = \|f\|_{L^1}$$

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- L^p improving estimate $A_r : L^{\frac{d+1}{d}}(\mathbb{R}^d) \to L^{d+1}(\mathbb{R}^d)$ adds

$$A_r : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$$

if and only if

$\left(\frac{1}{p}, \frac{1}{q}\right)$ is within the closed triangle $(0,0)$, $(1,1)$, $\left(\frac{d}{d+1}, \frac{1}{d+1}\right)$.

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- Sobolev bounds, such as

$$A_r : L^2_s(\mathbb{R}^d) \to L^2_{s+\frac{d-1}{2}}(\mathbb{R}^d)$$

Sobolev bounds show the density is bounded

- Key stationary phase estimate for Sobolev bound

$$|\widehat{\sigma}(\xi)| = \left| \int e^{-2\pi i y \cdot \xi} d\sigma(y) \right| \lesssim |\xi|^{-\frac{d-1}{2}}$$

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- Yields boundedness of the density if $\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{2}$

$$\delta(\mu_\epsilon)(r) \lesssim_r \|A_r(\mu_\epsilon)\|^{1/2}_{L^2_{\frac{d-1}{4}}} \|\mu_\epsilon\|^{1/2}_{L^2_{-\frac{d-1}{4}}} \leq \|\mu_\epsilon\|_{L^2_{-\frac{d-1}{4}}} = I_{\frac{d+1}{2}}(\mu_\epsilon)$$

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- $1 = \delta(\mu)(D(E)) = \int_{D(E)} \delta(\mu)(r) dr \leq \|\delta(\mu)\|_{L^\infty} \mathcal{L}(D(E))$

Results on the Falconer distance problem

- First results
  - Falconer $\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{2} \implies \mathcal{L}(D(E)) > 0$
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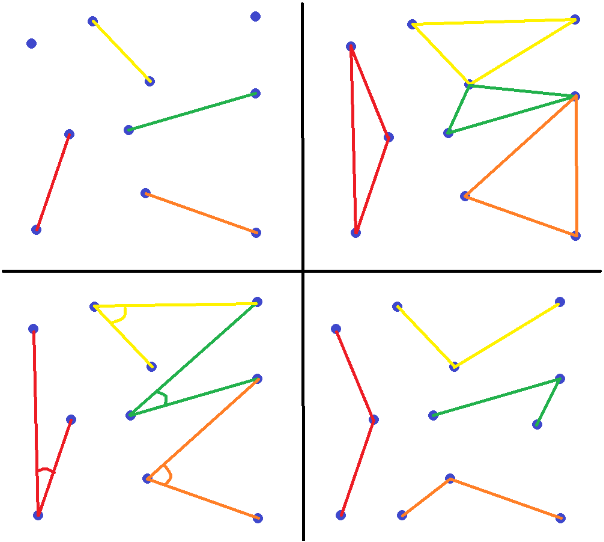
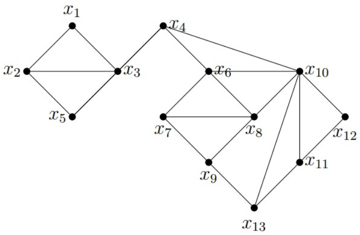
    - $\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{4} - \frac{1}{8d+4}$ in $\mathbb{R}^d$, $d \geq 3$

Many interesting point configurations



A road map



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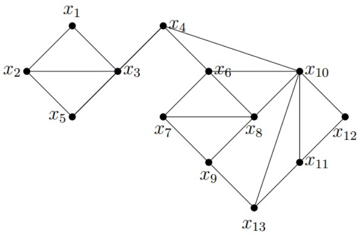

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$$\Lambda_G^{\epsilon}(f_1, \ldots, f_n) = \int \cdots \int \prod_{\{(i,j):1 \le i < j \le n; E(i,j)=1\}} \sigma^{\epsilon}(|x_i - x_j|) \prod_{i=1}^{n} f_i(x_i) dx_i$$

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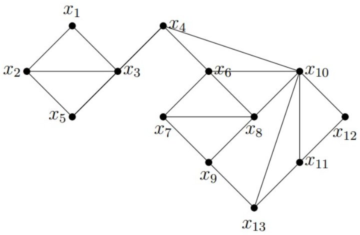

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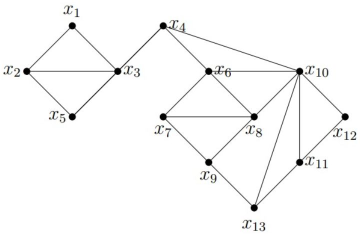

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- Can we even get L^p improving ones?

Regularly realizable distance configuration graphs

- G graph with with vertices $\{1, \ldots, n\}$ and edge set E
- Let $F : \mathbb{R}^{2n} \to \mathbb{R}^{|E|}$ be the map

$$F(x_1, \ldots, x_n) = (|x_i - x_j|)_{ij} \qquad \text{for } \{i, j\} \in E,\ i < j$$

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- Examples: Trees and Triangles
- A non-example: Cycle on 4 vertices.

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- A non-example: Cycle on 4 vertices.
- Regularly realizable generalizes a previous notion of a locally infinitesimally rigid configuration due to Chatzikonstantinou, Iosevich, Mkrtchyan and Pakianathan.

L^p improving estimates are possible

$$\Lambda_G^\epsilon(f_1,\dots,f_n) = \int\cdots\int \prod_{\{(i,j):1\le i<j\le n; E(i,j)=1\}} \sigma^\epsilon(|x^i - x^j|) \prod_{i=1}^n f_i(x^i)dx^i$$

Theorem (Iosevich, P, Wyman, Zhai)

Let G be a connected graph on $n \geq 2$ vertices which is regularly realizable in $\mathbb{R}^2$. Then, the multilinear form Λ_G^ϵ is bounded uniformly in ϵ on $L^{p_1}(\mathbb{R}^2)\times\cdots\times L^{p_n}(\mathbb{R}^2)$ for all $(\frac{1}{p_1},\dots,\frac{1}{p_n})$ contained in the convex hull of the points

$$\{e_1,\dots,e_n\} \cup \left\{\frac{2}{3}e_i + \frac{2}{3}e_j\right\}_{\{i,j\}\in E}.$$

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- This paradigm was introduced and studied in the case of some particular small graphs by Bhowmick, Iosevich, Koh and Pham in the finite field setting.

A Falconer type problem for triangles

- How large does $\dim_{\mathcal{H}}(E)$, for $E \subset \mathbb{R}^d$ compact, need to be to ensure that the set of triangles

$$D_\Delta(E) = \{(|x_1 - x_2|, |x_1 - x_3|, |x_2 - x_3|) : x_1, x_2, x_3 \in E\}$$

has positive three-dimensional Lebesgue measure?

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- Erdoğan and Iosevich conjecture for triangles in the plane

$$dim_{\mathcal{H}}(E) > \frac{3}{2} \text{ in } \mathbb{R}^2$$

- Only know the trivial restriction $\dim_{\mathcal{H}}(E) > \frac{d}{2}$ for $d \geq 3$.

Progress on the Falconer type problem for triangles

- Grafakos, Greenleaf, Iosevich, P.
  - $\dim_{\mathcal{H}}(E) > \frac{3}{4}d + \frac{1}{4}$ in $\mathbb{R}^d$
- Greenleaf, Iosevich, Liu, P.
  - $dim_{\mathcal{H}}(E) > \frac{8}{5}$ in $\mathbb{R}^2$
  - $dim_{\mathcal{H}}(E) > \frac{2d^2}{3d-1}$ in $\mathbb{R}^d$ when $d \geq 2$

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- Erdoğan, Hart, Iosevich
  - $\dim_{\mathcal{H}}(E) > \frac{1}{2}d + \frac{3}{2}$ in $\mathbb{R}^d$
- Iosevich, Pham, Pham, Shen
  - $\dim_{\mathcal{H}}(E) > \frac{1}{2}d + 1$ in $\mathbb{R}^d$

Mattila-Sjölin theorems for triangles

Theorem (P, Romero Acosta)

Let $E \subset \mathbb{R}^d$*,* $d \geq 4$*, be compact. If* $\dim_{\mathcal{H}}(E) > \frac{2}{3}d + 1$ *then* $D_\Delta(E)$ *has non-empty interior.*

- View $D_\Delta(E)$ from side-angle-side.
- Builds on work of Iosevich and Liu.

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- Later matched by Greenleaf, Iosevich and Taylor.

Mattila-Sjölin theorems for triangles

Theorem (P, Romero Acosta)

Let $E \subset \mathbb{R}^d$, $d \geq 4$, *be compact. If* $\dim_{\mathcal{H}}(E) > \frac{2}{3}d + 1$ *then* $D_\Delta(E)$ *has non-empty interior.*

- View $D_\Delta(E)$ from side-angle-side.
- Builds on work of Iosevich and Liu.
- Later matched by Greenleaf, Iosevich and Taylor.

Theorem (P, Romero Acosta)

Let $E \subset \mathbb{R}^3$ *be compact. If* $\dim_{\mathcal{H}}(E) > \frac{23}{8}$ *then* $D_\Delta(E)$ *has non-empty interior.*

- Classic side-side-side viewpoint.
- Builds on work of Iosevich and Magyar.
- Extends to simplexes in higher dimensions.

To conclude

- Falconer distance problem $\leftrightarrow$ Spherical averaging operator

- More complicated configurations
  - Interesting operators
  - Interesting point configuration problems

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- More complicated configurations
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- Thank you!

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