A short survey of integer tilings

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Based on joint work with Benjamin Bruce,
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Tiling the integers with translates of one finite tile: an introduction

Tiling the integers with translates of one finite set

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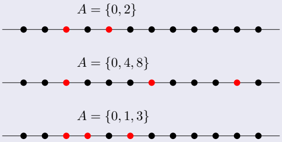

$A = \{0, 2\}$ and $A = \{0, 4, 8\}$ tile $\mathbb{Z}$; $A = \{0, 1, 3\}$ does not.

How to determine whether a given A tiles the integers?

Periodicity and reductions

Periodicity

All tilings of $\mathbb{Z}$ by a finite set A are periodic. Reduces the problem to tilings of finite cyclic groups $A \oplus B = \mathbb{Z}_M$ with addition mod M. (Newman)

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Prime factors reduction

We may assume that M has the same prime factors as $|A|$. (Coven-Meyerowitz, based on a theorem of Tijdeman)

Basic tools:

- Chinese Remainder Theorem (provides a multidimensional geometric representation),
- Mask polynomials of sets,
- Cyclotomic polynomials.

Geometric representation via Chinese Remainder Theorem

Suppose that $M = \prod_{i=1}^{d} p_i^{n_i}$, p_i distinct primes, $n_i \geq 1$. We may represent the cyclic group $\mathbb{Z}_M = \{0, 1, \ldots, M-1\}$ mod M as

$$\mathbb{Z}_M = \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{{p_d}^{n_d}}$$

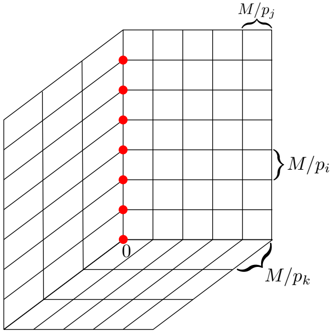
$$x = x_1 M/p_1^{n_1} + \cdots + x_d M/p_d^{n_d}$$

Geometrically, this is a d-dimensional periodic lattice with multiple scales. It will be important that the periods in different directions are powers of distinct primes.

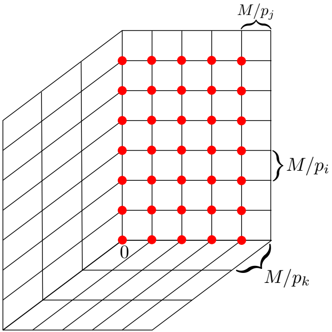
Geometric representation of sets

Numbers $a \in \mathbb{Z}_M$ are represented as lattice points.

$A = \{0, M/p_i, 2M/p_i, \ldots, (p_i - 1)M/p_i\}$



$A = \{x \in \mathbb{Z}_M : M/p_ip_j | x\}$



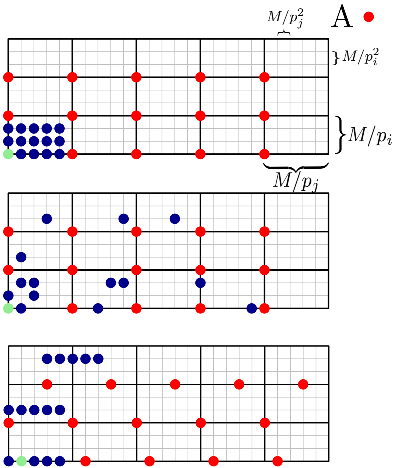
In this representation:

With $M = \prod_{i=1}^{d} p_i^{n_i}$, we represent $\mathbb{Z}_M$ as a d-dimensional lattice

$$\mathbb{Z}_M = \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_d{}^{n_d}}.$$

Then $A \oplus B = \mathbb{Z}_M$ is a tiling of that lattice (note the periodicity conditions!)

Examples of tilings



Cyclotomic polynomials:

The s-th *cyclotomic polynomial* is the unique monic, irreducible polynomial $\Phi_s(X)$ whose roots are the primitive s-th roots of unity. Alternatively, Φ_s can be defined inductively via

$$X^n - 1 = \prod_{s|n} \Phi_s(X).$$

To initialize, $X - 1 = \Phi_1(X)$, and then...

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$$\begin{aligned} X^6 - 1 &= (X^3-1)(X^3+1) \\ &= \underbrace{(X-1)}_{\Phi_1}\underbrace{(X^2+X+1)}_{\Phi_3}\underbrace{(X+1)}_{\Phi_2}\underbrace{(X^2-X+1)}_{\Phi_6} \end{aligned}$$

Polynomial formulation of tiling

We may assume that $A, B \subset \{0, 1, \dots\}$. Define the *mask polynomials*

$$A(X) = \sum_{a \in A} X^a, \ B(X) = \sum_{b \in B} X^b.$$

Then $A \oplus B = \mathbb{Z}_M$ is equivalent to

$$A(X)B(X) = 1 + X + \cdots + X^{M-1} \mod (X^M - 1).$$

Equivalently, $|A||B| = M$ and each $\Phi_s(X)$ with $s|M$, $s \neq 1$, divides at least one of $A(X)$ and $B(X)$.

The Coven-Meyerowitz tiling conditions

Coven-Meyerowitz tiling conditions

C-M (1998) proposed conditions (T1), (T2) on the distribution of these cyclotomic factors.

- (T1) is a relatively simple counting condition.
- (T2) is a deeper structural condition, equivalent to saying that each factor in the tiling may be replaced by a "standard" tile with a nice lattice-like structure.

Proved (T1) for all tiles, (T2) for tiles with $|A| = p^\alpha q^\beta$, p, q prime.

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The **C-M conjecture** (that (T2) holds for all finite tiles) is the main open problem in the theory of integer tilings.

Coven-Meyerowitz tiling conditions

Example 1.

Suppose $\Phi_2\Phi_3|A$, and A has no other prime power cyclotomic divisors. Then A tiles $\mathbb{Z}$ if and only if

$$|A| = 6 \text{ and } \Phi_6|A$$

(in other words, $A \equiv \{0,1,2,3,4,5\} \bmod 6$) – proved in C-M

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Example 2.

Suppose $\Phi_2\Phi_3\Phi_5|A$, no other prime power cyclotomic divisors. (T1)-(T2) say that if A tiles $\mathbb{Z}$, then

$$A \equiv \{0, 1, 2, \ldots, 29\} \bmod 30.$$

We do not know whether this is true.

Application: Minimal tiling period

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Assume A tiles $\mathbb{Z}$, and let $D = \max(A) - \min(A)$. Given a tiling $A \oplus T = \mathbb{Z}$, what is the minimal period of that tiling? What is the minimal tiling period among *all possible tilings* of $\mathbb{Z}$ by A?

- Let $A = \{0, 10, 20\}$. Then $A \oplus \{0, 1, \ldots, 9\} = \mathbb{Z}_{30}$ and the minimal period of this tiling is 30.

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- Let $A = \{0, 10, 20\}$. Then $A \oplus \{0, 1, \dots, 9\} = \mathbb{Z}_{30}$ and the minimal period of this tiling is 30.
- But A is also a complete set of residues mod 3. Therefore $A \oplus \{0\} = \mathbb{Z}_3$, and the minimal tiling period of A (minimized over all possible tilings) is 3.

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- There exist tilings of period at least $e^{c \log^2 D / log \log D}$ (Steinberger, improving on earlier work by Kolountzakis). The tiling period M has prime factors that $|A|$ does not have.

Application: minimal tiling period

Łaba-Zakharov 2024: assume A tiles the integers, and let $D = \max(A) - \min(A)$. Then:

- A admits a tiling of period at most $e^{c \log^2 D / log \log D}$. (Any tiling where M has the same prime factors as $|A|$ satisfies this.)
- For any $\epsilon > 0$ there exist tilings of period at least $D^{3/2-\epsilon}$, with $|A|$ and M having the same prime factors.

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If A satisfies (T2), it admits a tiling with period at most $2D$.

T2 results

3-prime result (Łaba-Londner 2021-22)

Theorem. Suppose that $A \oplus B = \mathbb{Z}_M$, with $M = \prod_{i=1}^{3} p_i^2$. (This is the simplest case that cannot be reduced to two prime factors using C-M methods.) Then A and B both satisfy (T2).

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Additionally:

- The proof also provides a classification of all tilings of period $M = \prod_{i=1}^{3} p_i^2$.
- Partial results for more general M; to complete the proof, we also need geometric arguments specific to 3 primes, 2 scales. Might go wrong for many distinct prime factors.

Why are 3 prime factors more difficult?

Sands: If $A \oplus B = \mathbb{Z}_M$ and M has at most 2 distinct prime factors, then at least one of A, B is contained in a coset of a proper subgroup of $\mathbb{Z}_M$.

This can be used to set up an inductive argument.

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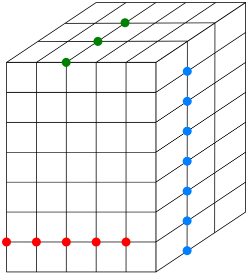
If M has 3 or more distinct prime factors, Sands's theorem no longer holds. We use a "fiber-shifting" example due to Szabó to demonstrate this.

Why are 3 prime factors more difficult?

Let $M = \prod p_i^{n_i}$. A fiber in the p_i direction is a translate of

$$F_i = \{0, M/p_i, \ldots, (p_i - 1)M/p_i\}.$$



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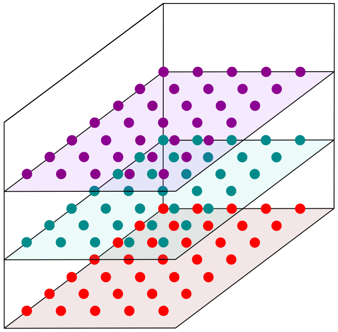

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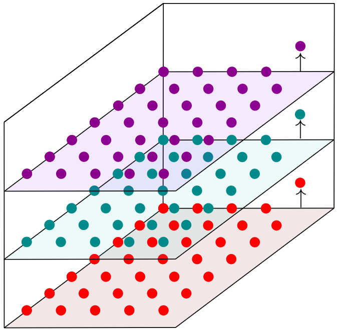

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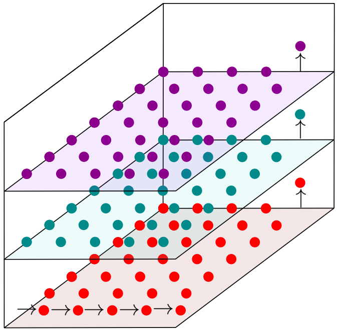

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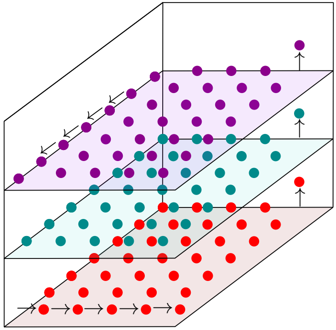
Example due to Szabó (1985)



Resolving tilings with 3 primes

To prove (T2) for $M = p_1^2 p_2^2 p_3^2$, we reverse this procedure:

- Find places where we think a fiber has been shifted.
- Find those fibers and shift them back.
- This reduces the tiling to one with a simpler structure.
- Repeat until (T2) is known.

A more recent tiling result (Łaba-Londner 2024)

We prove (T2) for both sets in $A \oplus B = \mathbb{Z}_M$, assuming that one of the prime factors of M is large compared to others. For example, (T2) holds for A and B if:

- $M = p_1^{n_1} p_2^{n_2} p_3^{n_3}$ for any $n_1, n_2, n_3 \in \mathbb{N}$, if $p_1 > p_2^{n_2 - 1} p_3^{n_3 - 1}$.

  (This includes $M = p_1^{n_1} p_2^2 p_3^2$ if $p_1 > p_2 p_3$.)
- $M = p_1^n p_2^2 p_3^2 p_4^2$ for any $n \in \mathbb{N}$, if $p_1 > p_2 p_3 p_4$.

Large prime result: sketch of proof

- Divisor sets and divisor exclusion
- Splitting for fibers
- Large prime implies splitting uniformity
- Splitting uniformity implies tiling reduction

Large prime result: divisor sets

Define $\mathrm{Div}(A) = \{(a - a', M) : \ a, a' \in A\}$, and similarly for B.

Divisor exclusion (Sands)

Let $A, B \subset \mathbb{Z}_M$. Then $A \oplus B = \mathbb{Z}_M$ if and only if $|A|\,|B| = M$ and

$$\mathrm{Div}(A) \cap \mathrm{Div}(B) = \{M\}.$$

Large prime result: splitting for fibers

Given a fiber $z + F_i$, consider the elements of A and B that tile that fiber:

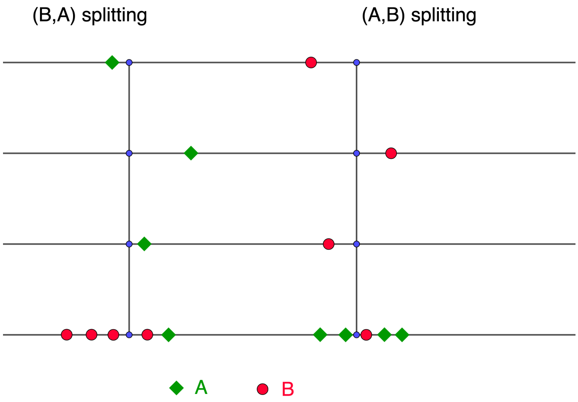
$$a_\nu + b_\nu = z + \nu M/p_i, \quad \nu = 0, 1, \ldots, p_i - 1.$$

Divisor exclusion implies that one of the following happens:

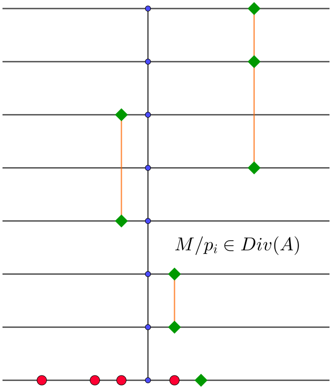
(a) $\forall \nu \neq \mu$, $p_i^{n_i} \mid a_\nu - a_\mu$ and $p_i^{n_i-1} \parallel b_\nu - b_\mu$,
(b) $\forall \nu \neq \mu$, $p_i^{n_i} \mid b_\nu - b_\mu$ and $p_i^{n_i-1} \parallel a_\nu - a_\mu$,

Splitting parity is (A, B) in (a), (B, A) in (b).

Large prime result: splitting for fibers



Large prime implies splitting uniformity



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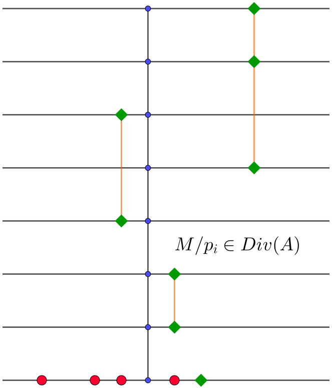

If one of the primes is large, (B, A) splitting implies $M/p_i \in \mathrm{Div}(A)$.

By divisor exclusion, cannot have that for both A and B. Therefore all fibers in the p_i direction split with the same parity.

Large prime result: splitting uniformity implies tiling reduction

If the splitting parity is uniform, we can apply the *slab reduction*:

- The tiling can be decomposed into p_i separate tilings of $\mathbb{Z}_{M/p_i}$.
- (T2) holds for the original tiling if and only if it holds for the smaller tilings.
- Proceed by induction until (T2) is known.

More general cases with many primes?

More primes?

General fact: high-dimensional tilings are complicated. For example:

Keller's conjecture for cube tilings

In any tiling of $\mathbb{R}^d$ by translates of the unit cube, there must be two cubes that share a full $(d-1)$-dimensional face.

- True for $d \leq 7$ (Perron; Brakensiek-Heule-Mackey--Narváez).
- False for $d \geq 8$ (Lagarias-Shor, Mackey).

Keller-type properties for integer tilings

Open question: Suppose that $A \oplus B = \mathbb{Z}_M$. Is it always true that $M/p \in \mathrm{Div}(A) \cup \mathrm{Div}(B)$ for some $p|M$ prime?

- Used in the large prime result.

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- Used in the large prime result.
- When $M = p_1^2 p_2^2 p_3^2$, we have the stronger result that one of A, B contains a fiber (used in our T2 proof).
- Bruce-Łaba 2024: we use counterexamples to Keller's conjecture to construct integer tilings with no fibers in either set. For T2 with more primes, new methods will be needed.

Thank you!

Coven-Meyerowitz theorem

Let $S_A = \{p^\alpha : \ \Phi_{p^\alpha}(X) | A(X)\}$. Consider the conditions:

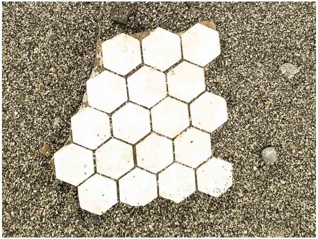
(T1) $A(1) = \prod_{s \in S_A} \Phi_s(1)$,

(T2) if $s_1, \ldots, s_k \in S_A$ *are powers of distinct primes, then* $\Phi_{s_1 \ldots s_k}(X)$ *divides* $A(X)$.

Then:

- if A satisfies (T1), (T2), then A tiles $\mathbb{Z}$;
- if A tiles $\mathbb{Z}$ then (T1) holds;
- if A tiles $\mathbb{Z}$ and $|A|$ has *at most two prime factors*, then (T2) holds.

Fuglede's spectral set conjecture



Conjecture: Assume that $\Omega \subset \mathbb{R}^n$ has non-zero and finite Lebesgue measure. Then Ω tiles $\mathbb{R}^n$ by translations if and only if $L^2(\Omega)$ admits an orthogonal basis of exponential functions (Ω is *spectral*)

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The conjecture, in its full generality, is false in dimensions $n \geq 3$ (Tao, Kolountzakis, Matolcsi, Farkas, Révész, Móra)

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But true in many special cases of interest:

- When the translation set is a lattice (Fuglede)
- Convex sets in $\mathbb{R}^n$ (Iosevich-Katz-Tao for $n = 2$; Greenfeld-Lev for $n = 3$, Lev-Matolcsi for $n \geq 4$).
- Finite group analogue, for groups with simple enough structure (Malikiosis, Kolountzakis, Iosevich, Mayeli, Pakianathan, Kiss, Somlai, Viser, Shi, Zhang...)

Connection to Coven-Meyerowitz tiling conditions

If (T2) holds for all tiles of $\mathbb{Z}_M$ for some M, then tiling implies spectrality in $\mathbb{Z}_M$. (Łaba)

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If C-M conjecture is true, then every bounded tile Ω of $\mathbb{R}$ is spectral. Follows by combining the above with earlier work by Lagarias-Wang.

Dutkay-Lai: If every spectral set $A \subset \mathbb{Z}$ satisfies (T1) and (T2), then every bounded spectral set in $\mathbb{R}$ tiles $\mathbb{R}$ by translations.

(T1) and (T2) imply spectrality

Let $A \subset \mathbb{Z}_M$. Let

$$\Lambda = \left\{ \sum_s \frac{k_s}{s} : \ k_s \in \{0, 1, \ldots, p-1\} \right\}$$

where s runs over all prime powers $s|M$ such that $\Phi_s|A$. Try

$$\{e^{2\pi i \lambda} : \ \lambda \in \Lambda + \mathbb{Z}\}$$

as an orthonormal basis for $L^2(A)$.

- If A satisfies T1, Λ has the "right" cardinality $|\Lambda| = |A|$.
- If A satisfies T2, then the given exponentials are pairwise orthogonal in $L^2(A)$.

New ideas needed for 3 prime factors:

- Suppose $\Phi_M|A$. Use this to establish initial structure, at first only on individual top-level grids.
- For "fibered grids", try to proceed by induction on scales.
- For "unfibered grids" (as in Szabó's example), use the irregularities to recover the rest of the tiling.
- To do all this, we had to develop new tools (box product, multiscale cuboids, saturating sets...). These tools can be applied to a range of other questions.