WITH STRINGS ATTACHED

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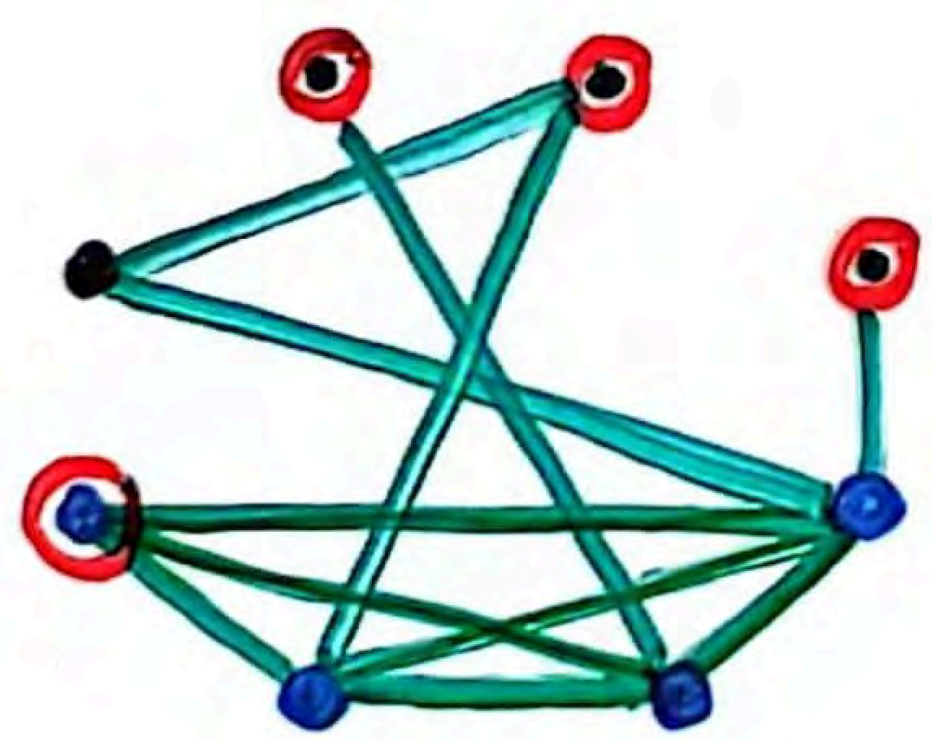
Heroes Square, Budapest, Hungary

HEROES' SQUARE

Theorem (Ramsey 1930, Erdös-Szekeres 1935)
Every graph of n vertices has a clique or an independent set of size $r(n)$, where

$$\left(\tfrac{1}{2}+o(1)\right)\log n \le r(n) \le \left(2+o(1)\right)\log n$$

$\frac{1}{2}+\varepsilon$ Campos-Griffith-Morris-Sahasrabudhe



HEROES' SQUARE

Conjecture (Erdös-Hajnal 1989)
For every graph H, there exists $\varepsilon = \varepsilon(H) > 0$ such that every n-vertex graph G with $H \notin G$ has a clique or an independent set of size $\geq n^{\varepsilon}$.

$e^{\sqrt{\log n}} \ll e^{\varepsilon \log n}$ Erdös-Hajnal 1989

$e^{\varepsilon \sqrt{\log n \log\log n}}$ Bucić-Nguyen-Scott-Seymour 2023

hereditary class of graphs $\mathcal{G}$:

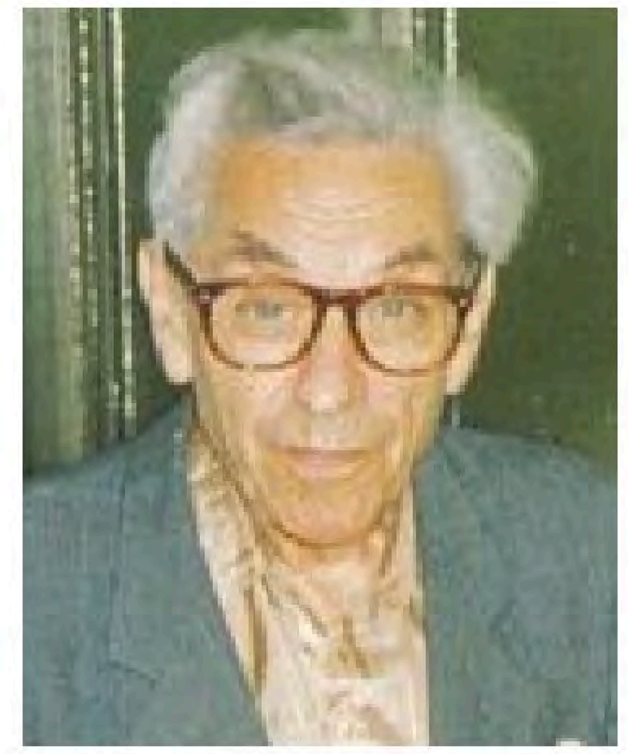
$G \in \mathcal{G}$, G' induced subgraph of $G \Longrightarrow G' \in \mathcal{G}$

HEROES' SQUARE

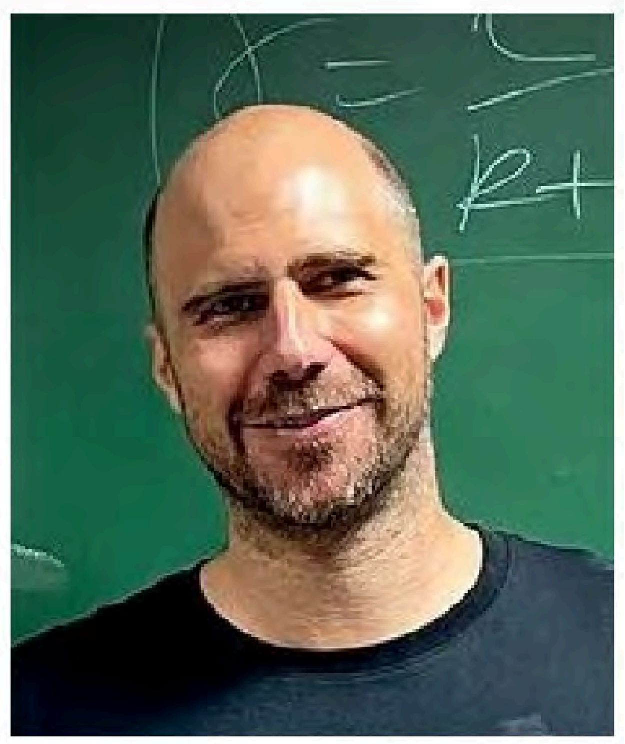

Bucic


Campos


Conlon


Erdős


Fox


Griffith

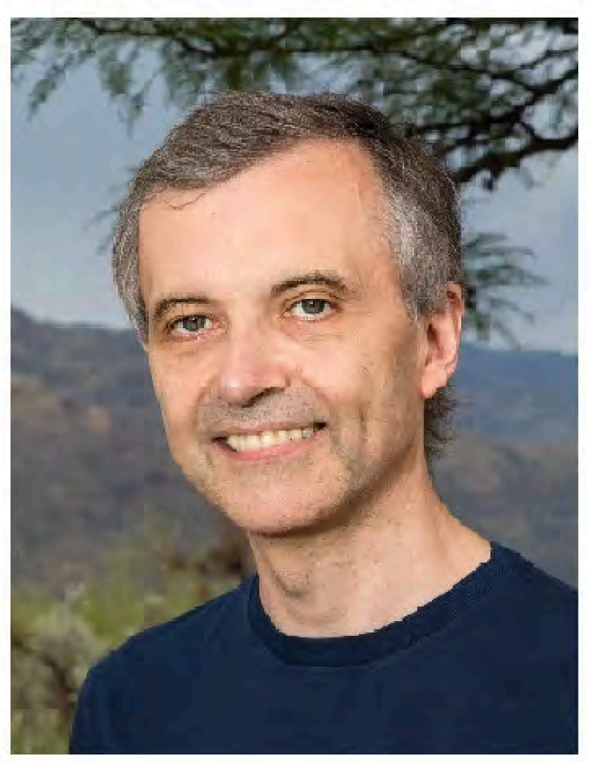

Hajnal


Morris

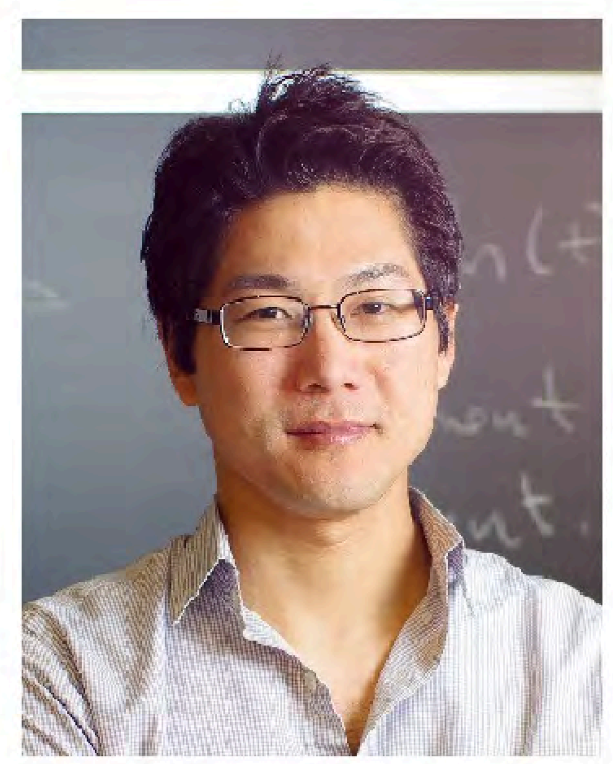

Nguyen


Ramsey


Sahasrabudhe


Scott


Seymour


Suk

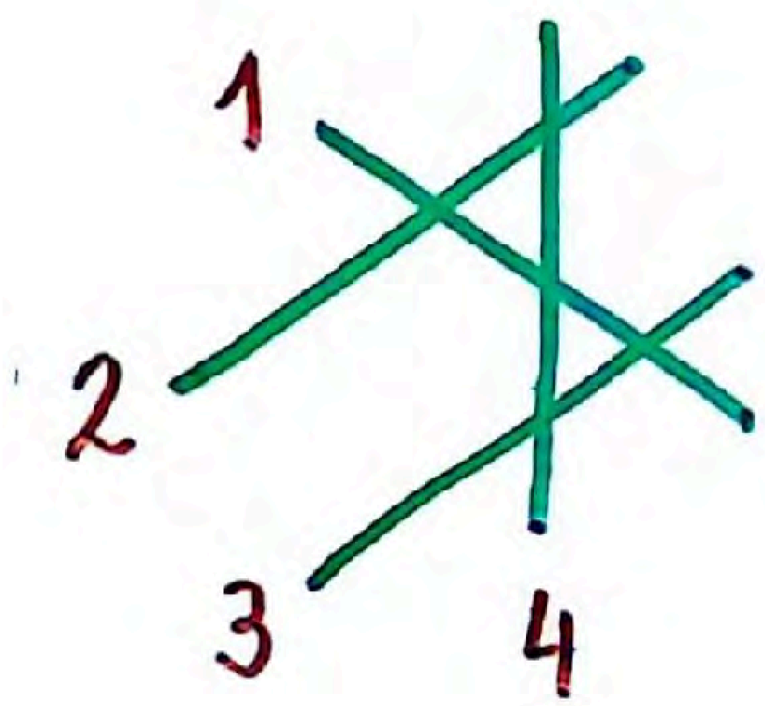

Szekeres

INTERSECTION GRAPHS

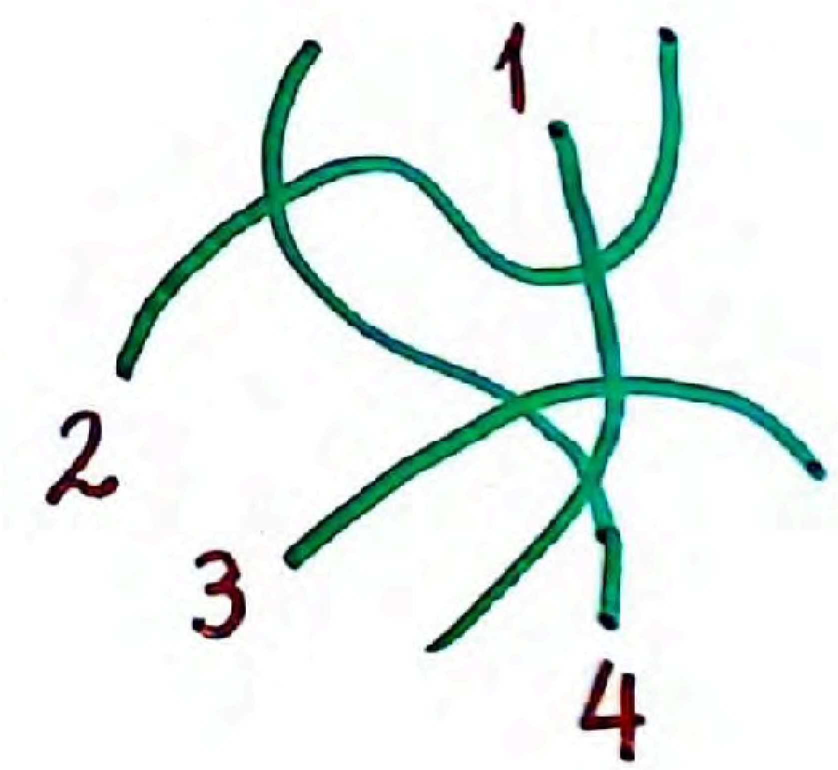
segment graphs

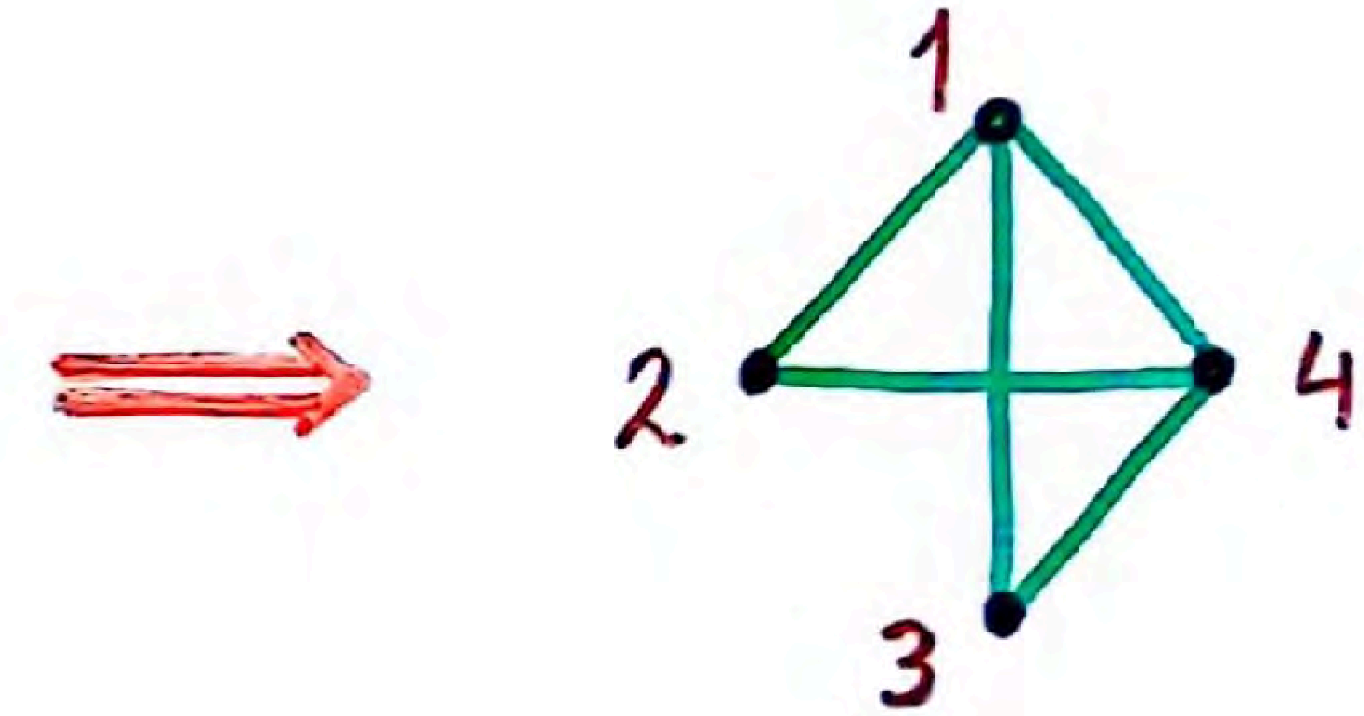
INTERSECTION GRAPHS

segment graphs



pseudosegment
intersection graphs
- any pair of curves intersect ≤ once





INTERSECTION GRAPHS

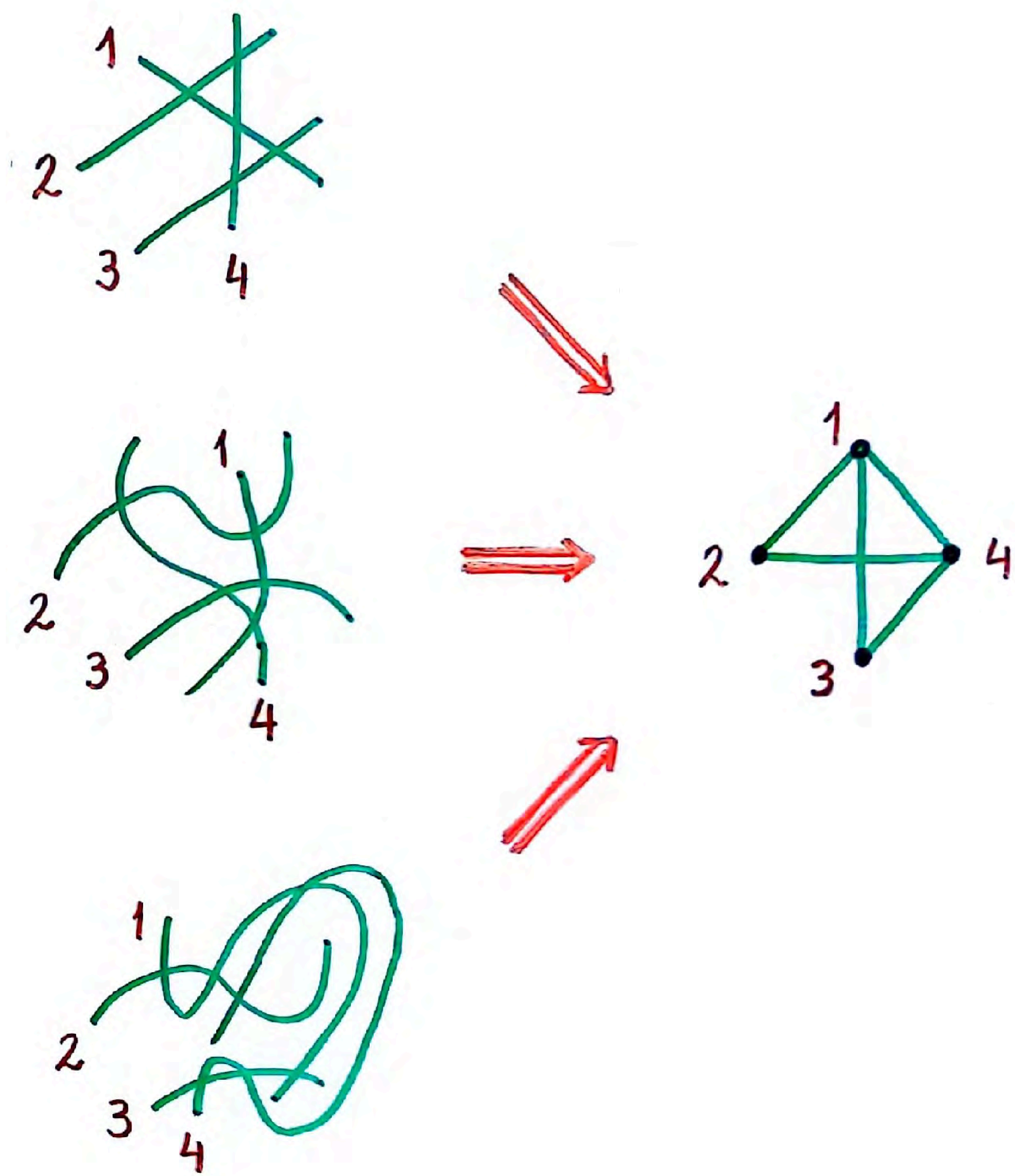
segment graphs

pseudosegment intersection graphs

- any pair of curves intersect $\leq$ once

string graphs

- intersection graphs of arbitrary curves (strings)



ERDŐS-HAJNAL PROPERTY

A class of graphs $\mathcal{G}$ has the Erdős-Hajnal property if there exists $\varepsilon = \varepsilon(\mathcal{G}) > 0$ such that every n-vertex graph $G \in \mathcal{G}$ has a clique or an independent set of size $\geq n^{\varepsilon}$.

Conjecture (Erdős-Hajnal 89)
Every nonempty hereditary class of graphs has the Erdős-Hajnal property.

Theorem (Larman-Matoušek-P.-Töröcsik 94)
Segment intersection graphs have the Erdős-Hajnal property (with $\varepsilon \geq 1/5$).

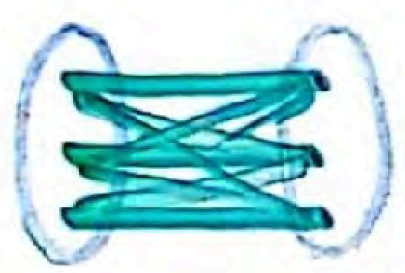
Problem. What is the best value of ε ???

$0.2 \leq \varepsilon < 0.404$ Kynčl 2012

STRONG ERDÖS-HAJNAL PROPERTY

A class of graphs $\mathcal{G}$ has the **strong** Erdös-Hajnal property if there exists $c = c(\mathcal{G}) > 0$ such that in every n-vertex $G \in \mathcal{G}$ one can find $A, B \subseteq V(G)$, $|A| = |B| \geq cn$ with the property that

$$A \times B \subseteq E(G) \quad \text{or} \quad (A \times B) \cap E(G) = \emptyset.$$



A B

or

A B

$\Longrightarrow$ Erdös-Hajnal property

Theorem (P.-Solymosi 2001)
Segment intersection graphs have the **strong** Erdös-Hajnal property.

semialgebraic graphs Alon-P.-Pinchasi-Radoičić-Sharir 2005

E-H PROPERTY $\neq$ STRONG E-H ?

Theorem (P.-G. Tóth : Comment on Fox news 2006)
String graphs (intersection graphs of strings) do **not** have the strong Erdös-Hajnal property.

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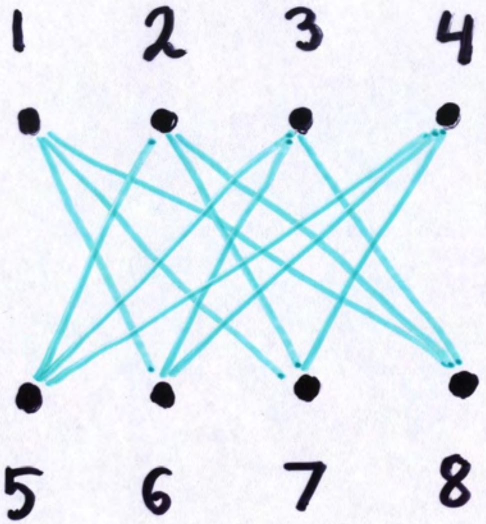
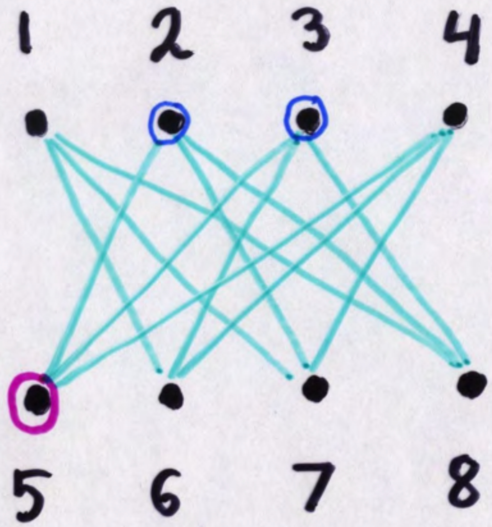
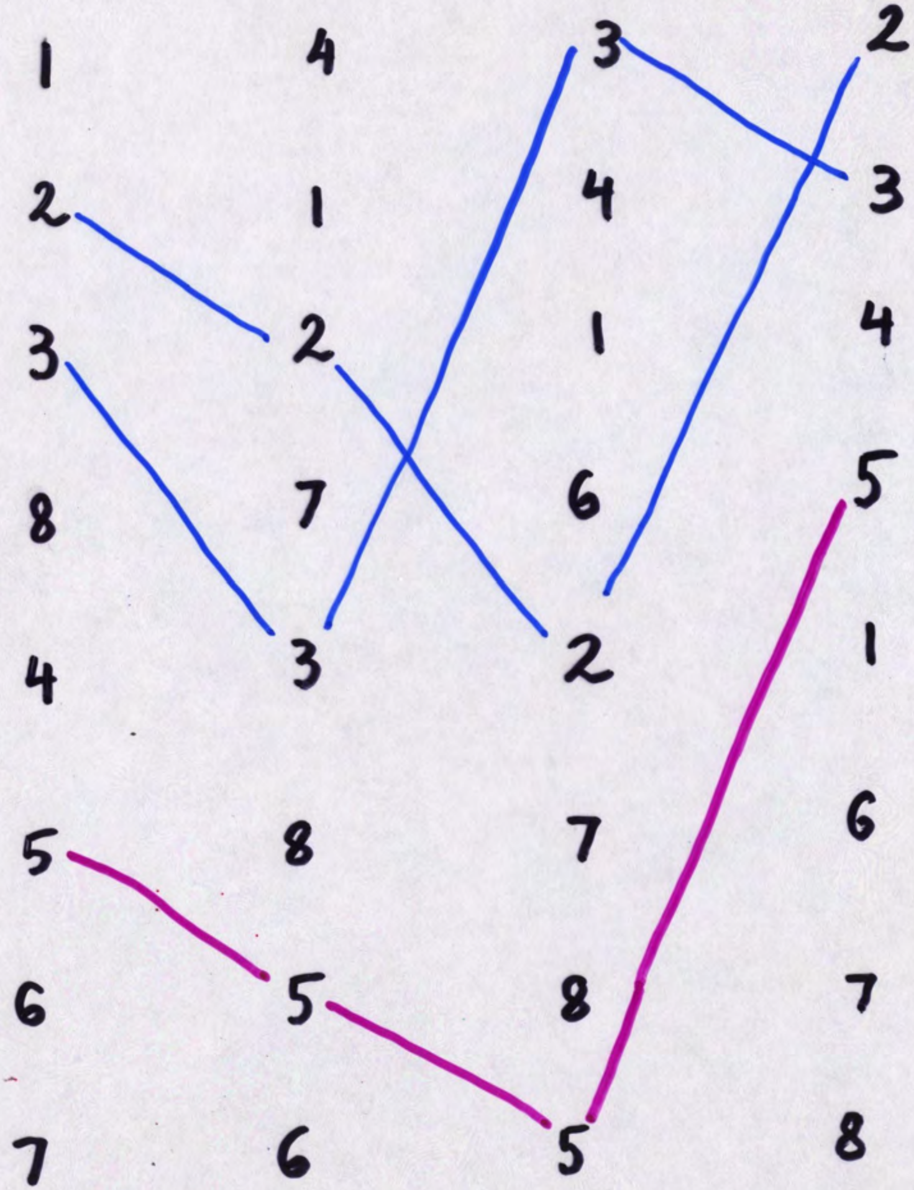
Theorem (Golumbic-Rotem-Urrutia 83, Lovász 83)
Every incomparability graph is a string graph, but not vice versa.



1	4	3	2
2	1	4	3
3	2	1	4
8	7	6	5
4	3	2	1
5	8	7	6
6	5	8	7
7	6	5	8


1 2 3 4
5 6 7 8


1 2 3 8 4 5 6 7
4 1 2 7 3 8 5 6
3 4 1 6 2 7 8 5
2 3 4 5 1 6 7 8

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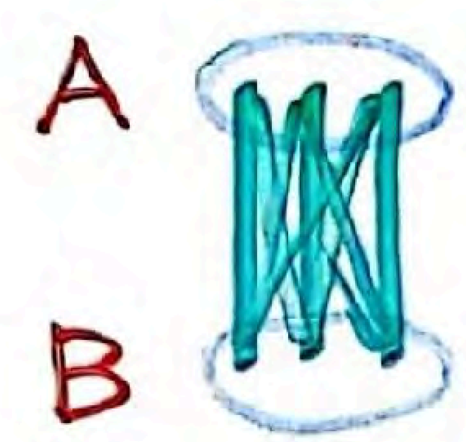
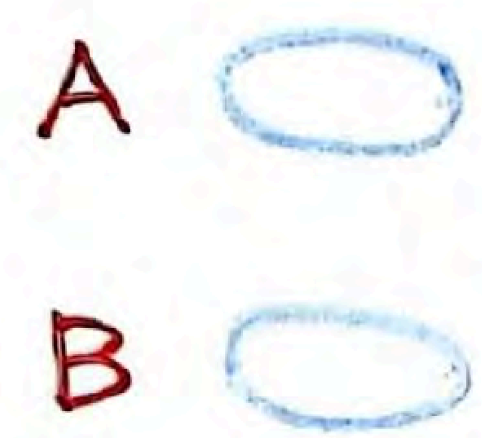
Theorem (Fox 2006) There is a partially ordered set P, $|P| = n$ with no disjoint subsets $A, B \subset P$, $|A| = |B| \geq \frac{n}{\log n}$ such that every $a \in A$ is comparable to every $b \in B$ or no $a \in A$ is comparable to any $b \in B$.

NO



and



E-H PROPERTY ≠ STRONG E-H

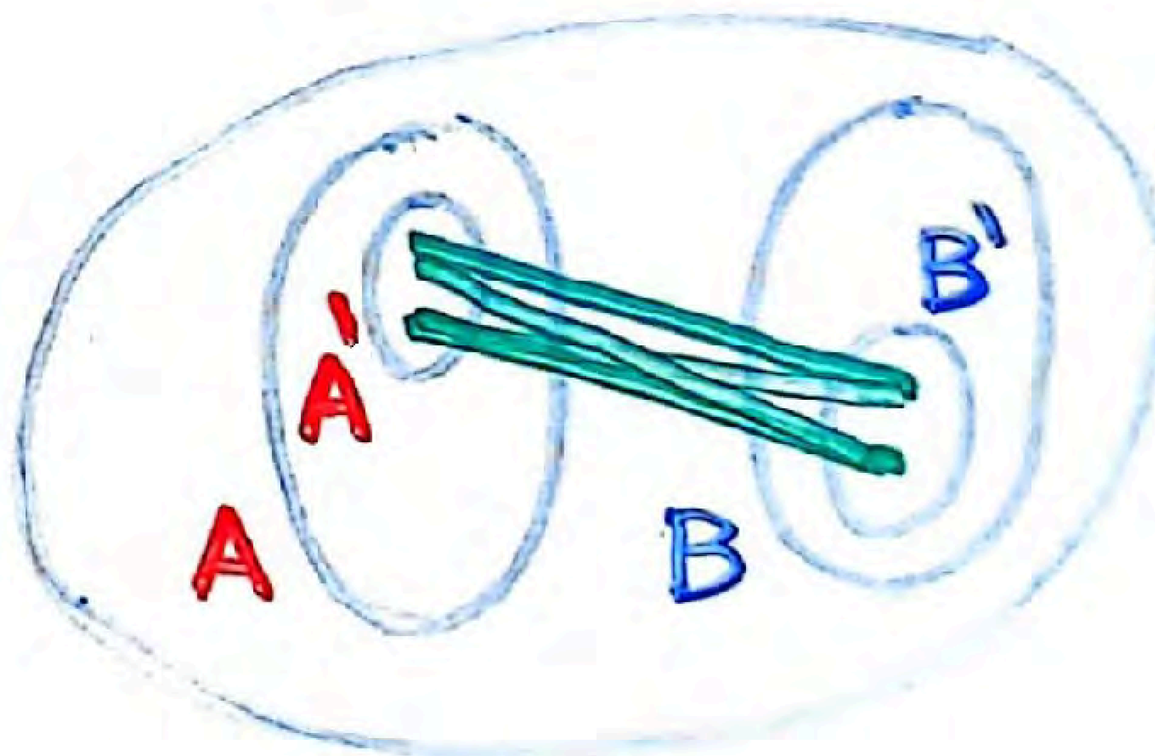
Theorem (P.-G. Tóth : Comment on Fox news 2006)
String graphs (intersection graphs of strings) do not have the strong Erdös-Hajnal property.

Theorem (Tomon 2023)
String graphs have the Erdös-Hajnal property.

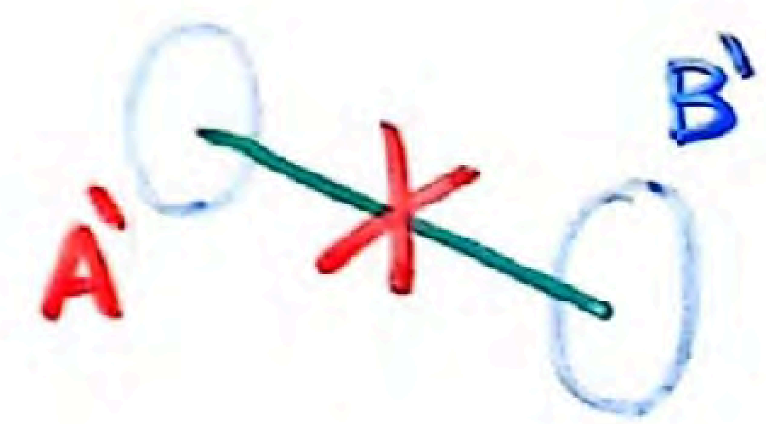
Even Stronger: 'Mighty' E-H Property

A class of graphs $\mathcal{G}$ has the mighty E-H property if there exists $c = c(\mathcal{G}) > 0$ with the property that for any $G \in \mathcal{G}$ and $A, B \subset V(G)$, one can find $A' \subseteq A$, $B' \subseteq B$ with $|A'| \geq c|A|$, $|B'| \geq c|B|$ such that the bipartite graph between A and B is complete or empty.

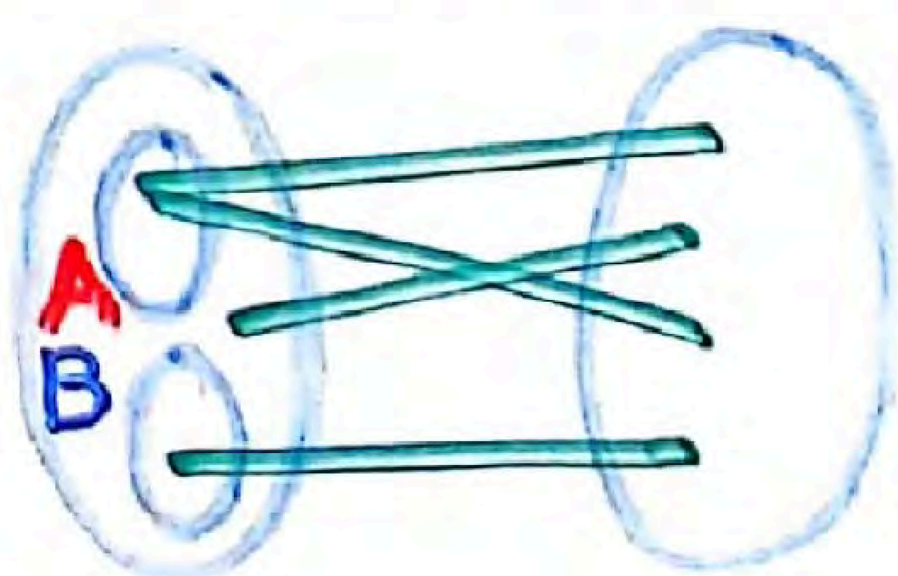
$\Longrightarrow$ strong E-H property



or



$\not\Longleftarrow$ $\mathcal{G}$ = class of bipartite graphs



(random)

MIGHTY E-H $\neq$ STRONG E-H PROPERTY

Theorem (P.-Solymosi 2001)
Segment intersection graphs have the mighty Erdős-Hajnal property.

Theorem (Fox-P.-Cs.Tóth 2010)
Intersection graphs of convex sets in the plane have the strong Erdős-Hajnal property, but not the mighty one.

Mighty E-H ≠ Strong E-H Property

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pseudosegments - collection of strings, any pair of which intersect $\leq$ once

Theorem (Fox-P.-Cs.Tóth 2010)
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mighty ??

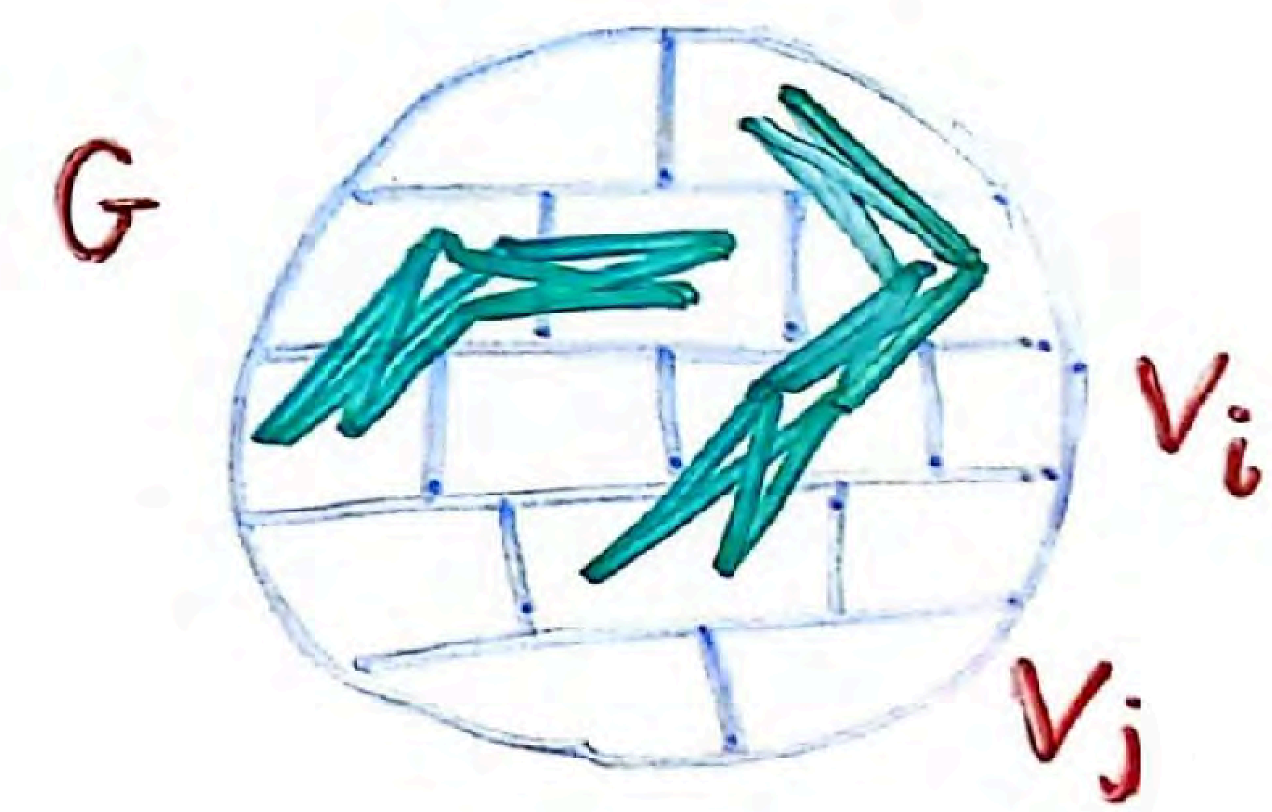
PSEUDOSEGMENTS

Theorem (Fox-P.-Suk 2024)
Pseudosegment intersection graphs have the mighty Erdős-Hajnal property.

$\Downarrow$

"Perfect" regularity lemma.

For every $\varepsilon > 0$, there exists $K = K(\mathcal{G}, \varepsilon)$ with the property that the vertex set of every $G \in \mathcal{G}$ can be partitioned into K equal parts $V_1 \cup V_2 \cup \dots \cup V_K$ such that all but $\leq \varepsilon K^2$ pairs (V_i, V_j) induce complete or empty bipartite graphs.



Conjecture. If $\mathcal{G}$ = pseudosegments,
$K = O(1/\varepsilon^c)$

PSEUDOSEGMENTS

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$\Downarrow \quad \Uparrow$

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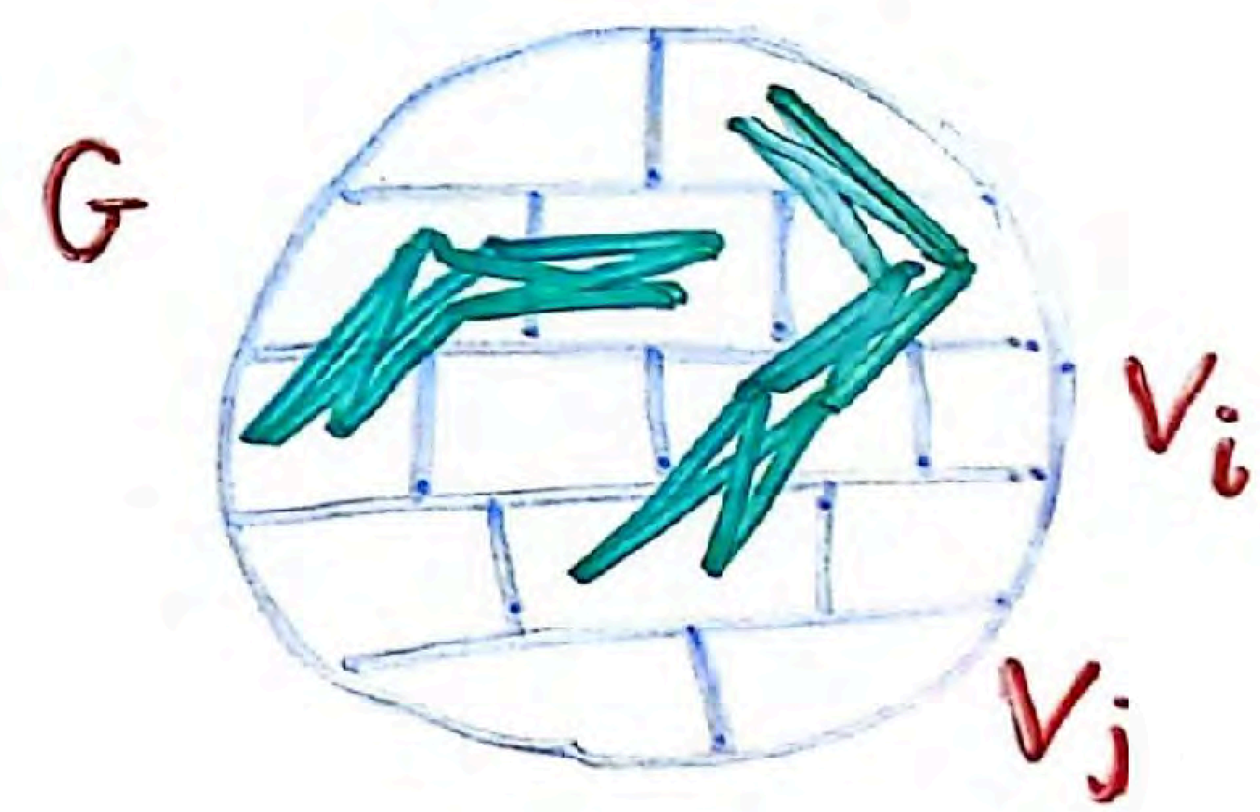

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$$\Updownarrow$$

"Perfect" density lemma.

For every $\varepsilon > 0$, there exists $c = c(\mathcal{G}, \varepsilon) > 0$ with the following property: For every $G \in \mathcal{G}$ and disjoint sets $A, B \subseteq V(G)$, $|A| = |B|$ with $\geq \varepsilon |A||B|$ edges between them, one can find $A' \subseteq A$, $B' \subseteq B$, $|A'| = |B'| \geq \varepsilon^c |A|$ that induce a complete bipartite subgraph in G.

The same is true for the class $\bar{\mathcal{G}} = \{\bar{G} : G \in \mathcal{G}\}$.

Theorem. Let $\mathcal{G}$ be a hereditary class of graphs. The following statements are equivalent.

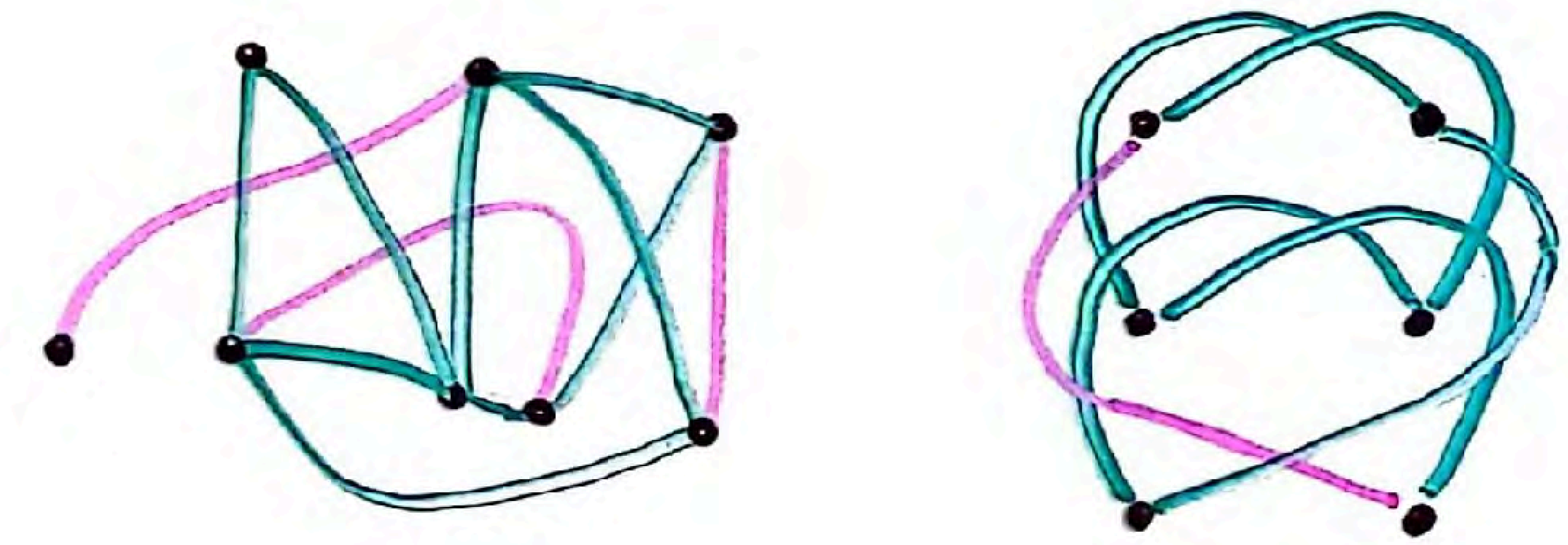
(i) $\mathcal{G}$ has the mighty Erdős-Hajnal property.

(ii) $\mathcal{G}$ satisfies the perfect regularity lemma.

(iii) $\mathcal{G}$ and $\overline{\mathcal{G}}$ satisfy the perfect density lemma.

Application of Density Theorem

simple drawing of a graph G - edges are pseudosegments



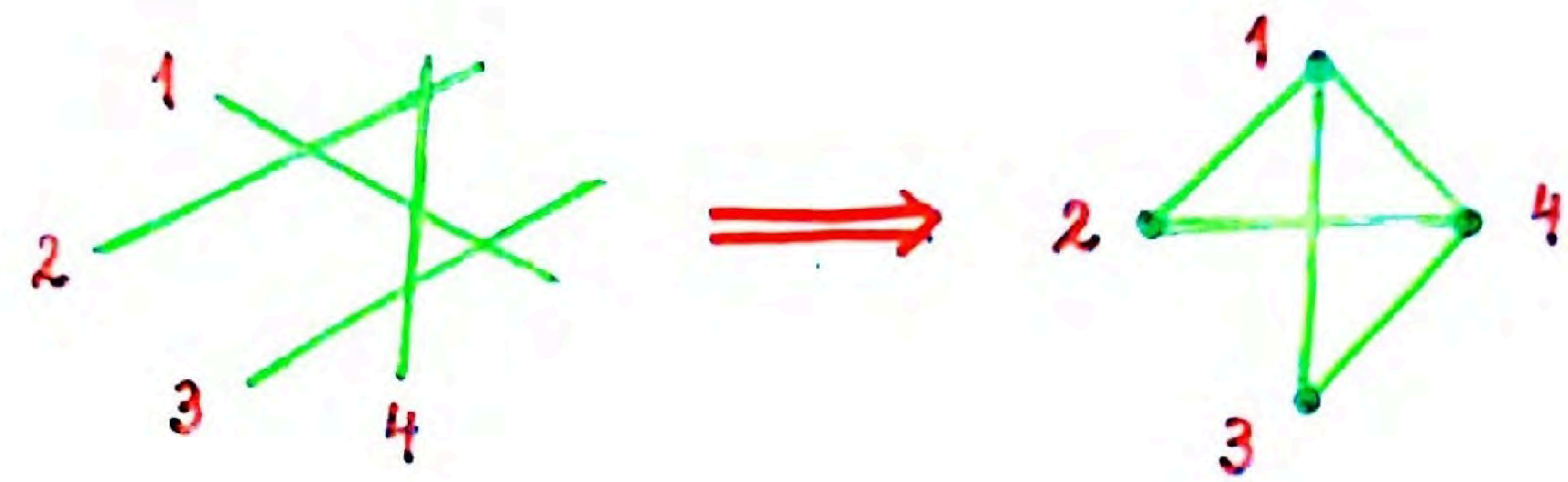
Theorem (P. - G. Tóth 2005)
If G has a simple drawing with no k pairwise disjoint edges, then $|E(G)| \le n(\log n)^{4k-8}$.

Theorem (Fox - P. - Suk 2024)
If G has a simple drawing with no k pairwise disjoint edges, then $|E(G)| \le n(\log n)^{O(\log k)}$.

Conjecture. $O_k(n)$??

ENUMERATION OF SEGMENT GRAPHS

segments



Theorem (P.-Solymosi 2001)

The number of segment intersection graphs on n labeled vertices is $\leq 2^{(4+o(1))n\log n}$.

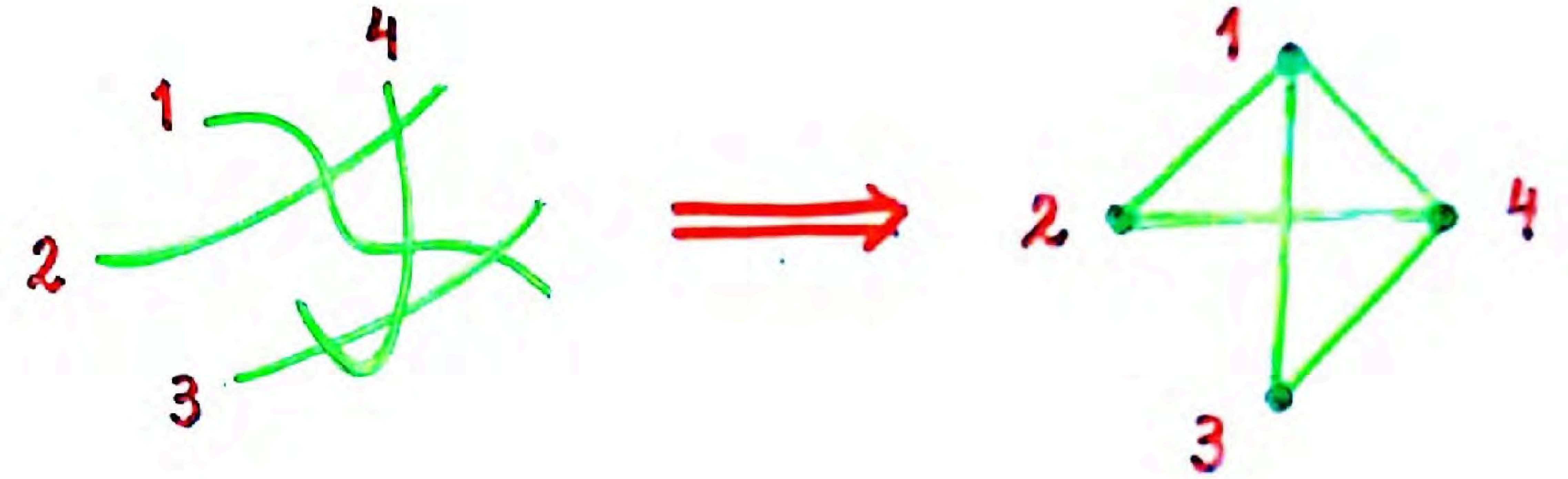
$$\ll 2^{\binom{n}{2}}$$

This bound is asymptotically tight.

Fox; Sauermann 2021

Enumeration of String Graphs

"strings"
(arcs)



string graph

Theorem (P.-Tóth 2002)

The number of labeled string graphs on n vertices is

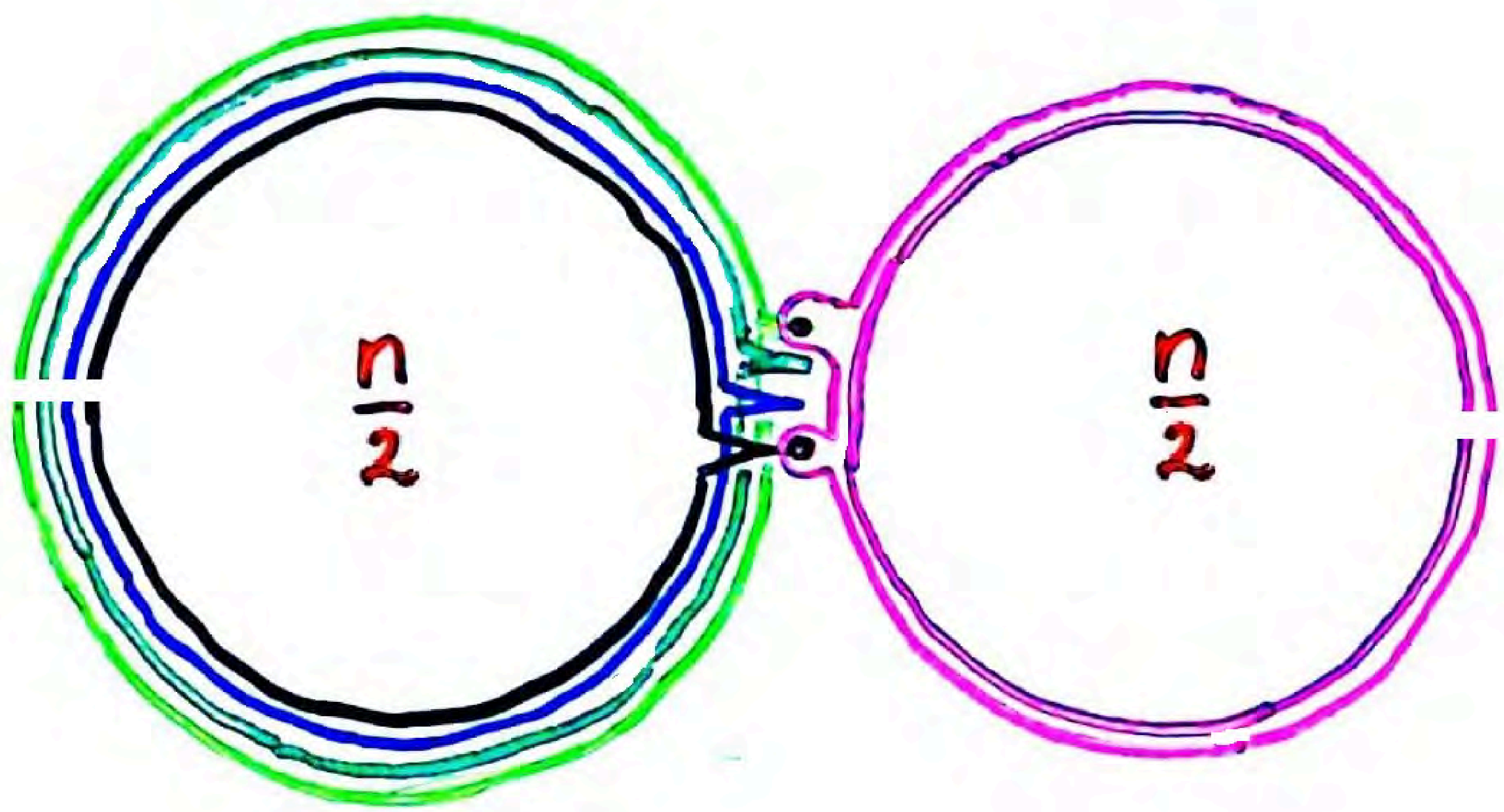
$$2^{\left(\frac{3}{4}+o(1)\right)\binom{n}{2}}$$

Theorem (P.-Tóth 2002)

The number of intersection graphs of n strings, any pair of which cross $\leq d$ times, is

$$2^{o(n^2)}$$

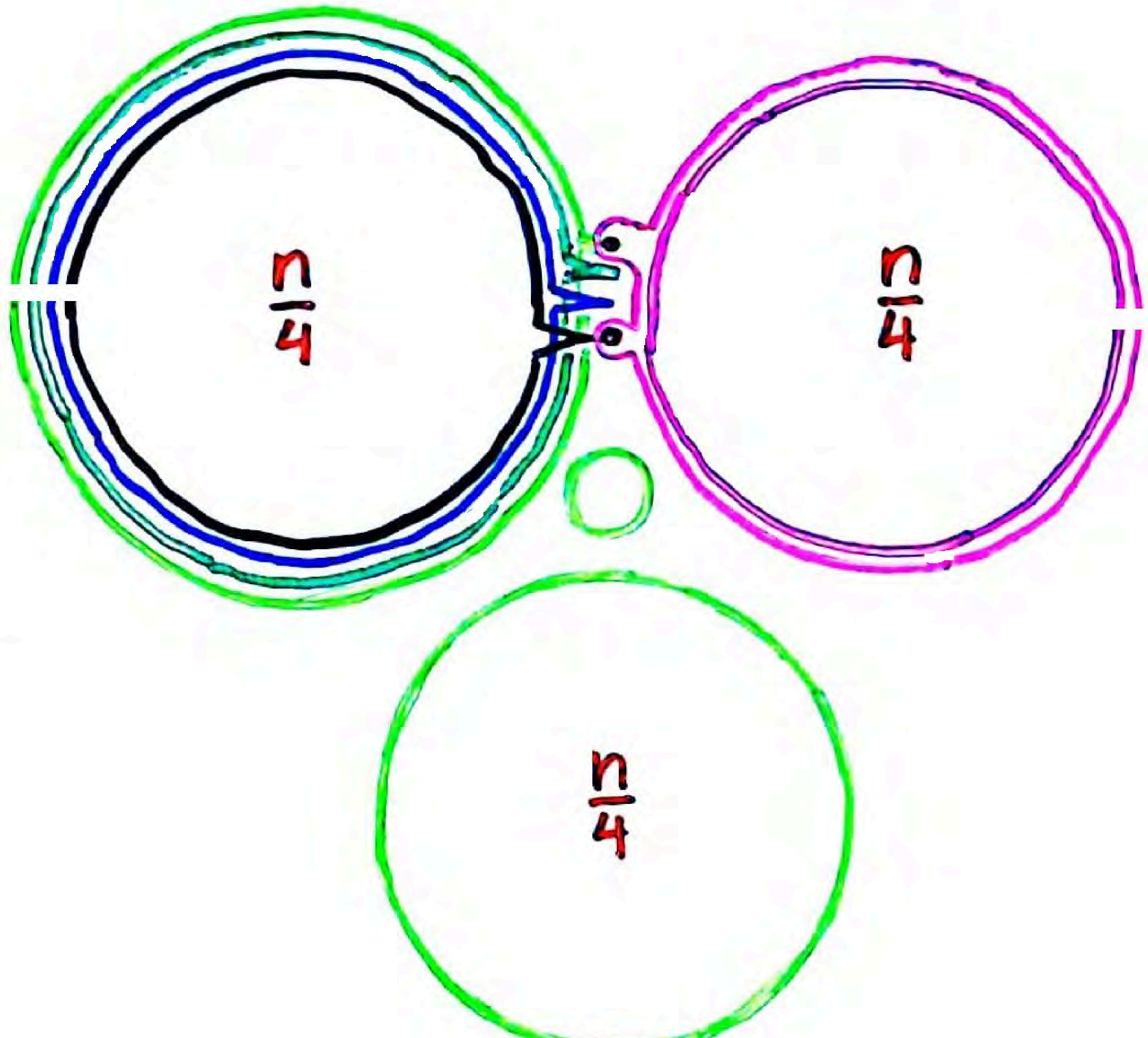
STRING GRAPHS – A CONSTRUCTION



$$\left(\frac{n}{2}\right)^2 = \frac{n^2}{4}$$

string graphs on n vertices $\geq 2^{\frac{n^2}{4}}$

STRING GRAPHS – A CONSTRUCTION



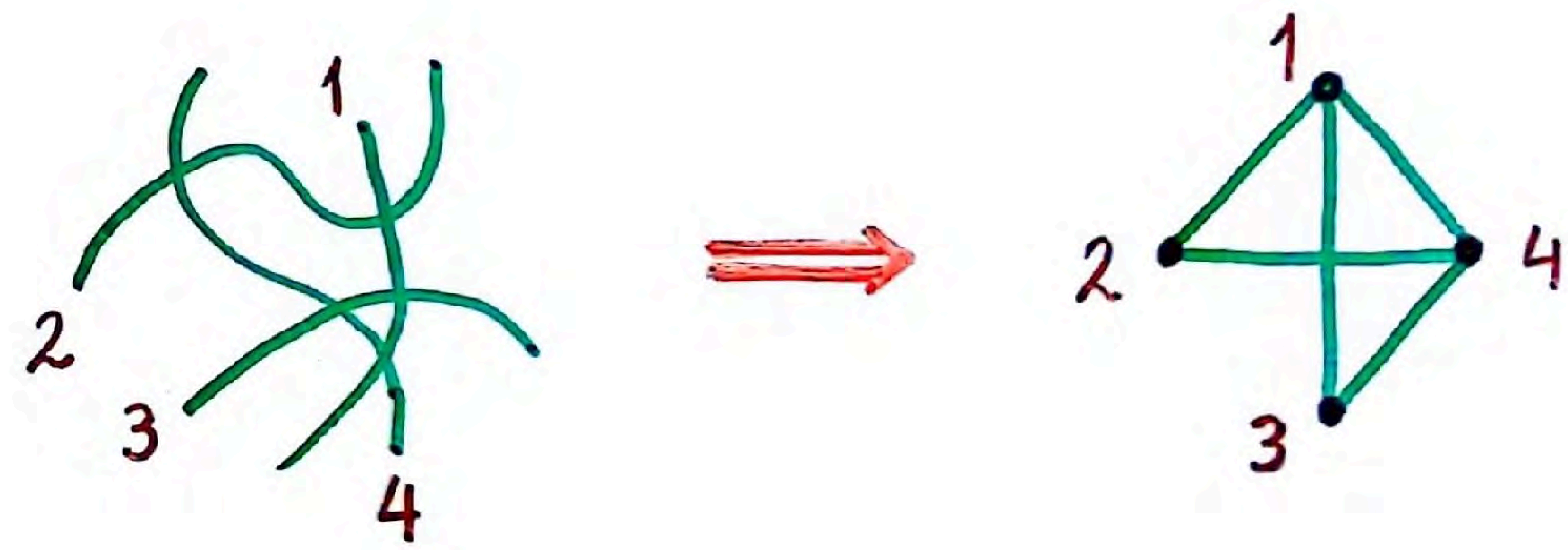
$$\binom{4}{2}\left(\frac{n}{4}\right)^2 = \frac{3}{4}\frac{n^2}{2}$$

$\#$ string graphs on n vertices $\geq 2^{\frac{3}{4}\frac{n^2}{2}}$

Enumeration of Pseudosegment Graphs

pseudosegment intersection graphs

- any pair of curves intersect $\leq$ once

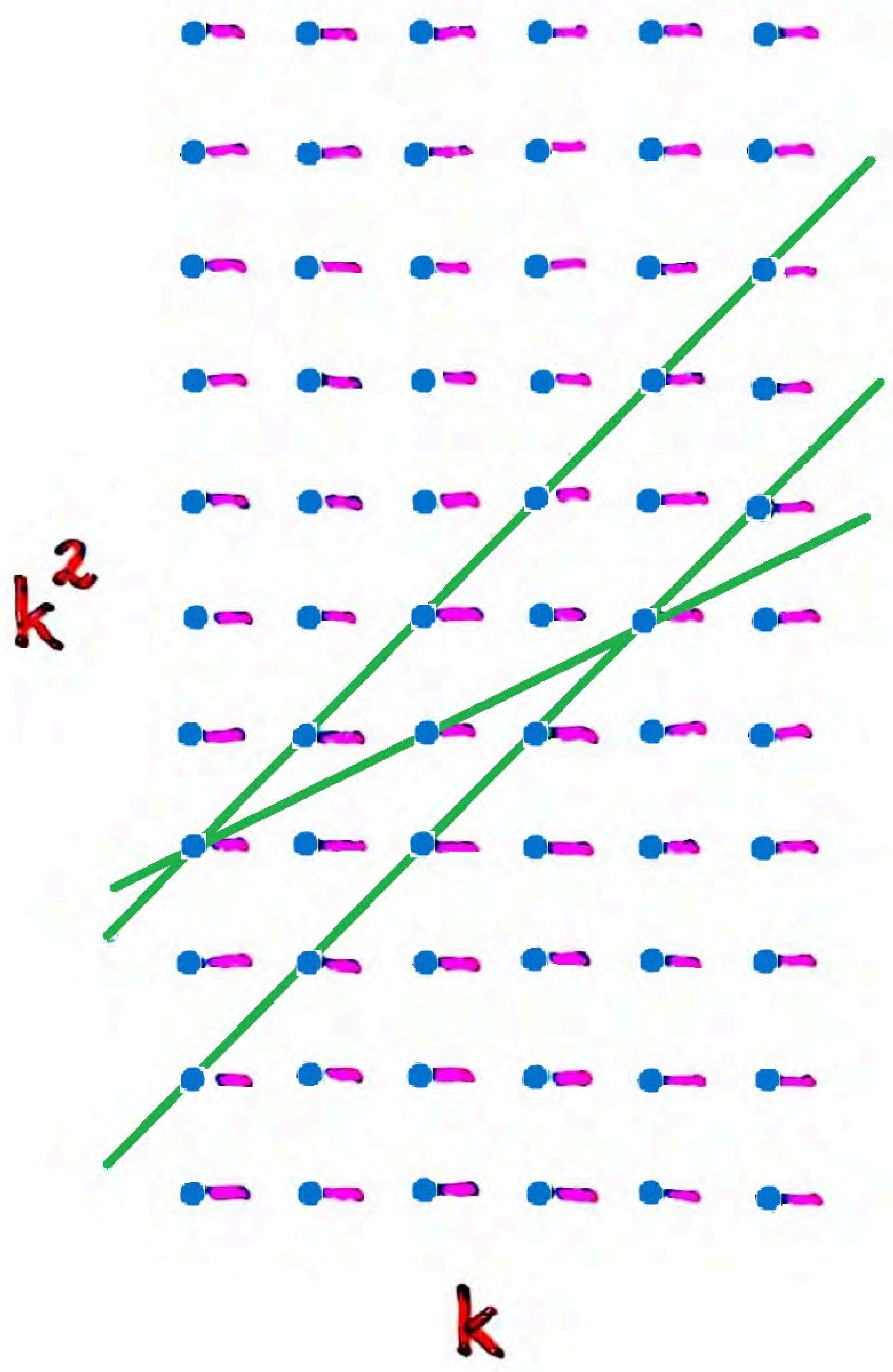


Theorem (Fox-P.-Suk 2022)

The number of pseudosegment intersection graphs on n labeled vertices is

$$\geq 2^{cn^{4/3}} \gg 2^{(4+o(1))n\log n}$$

A Construction with Pseudosegments



$n = k^3$ short segments

$n = k^3$ lines

$y = ax + b \quad (1 \le a \le k,\ 1 \le b \le k^2)$

$cnk = cn^{4/3}$ incidences

A Construction with Pseudosegments

$n = k^3$ short segments

$n = k^3$ pseudolines

$y = ax + b \quad (1 \leq a \leq k,\ 1 \leq b \leq k^2)$

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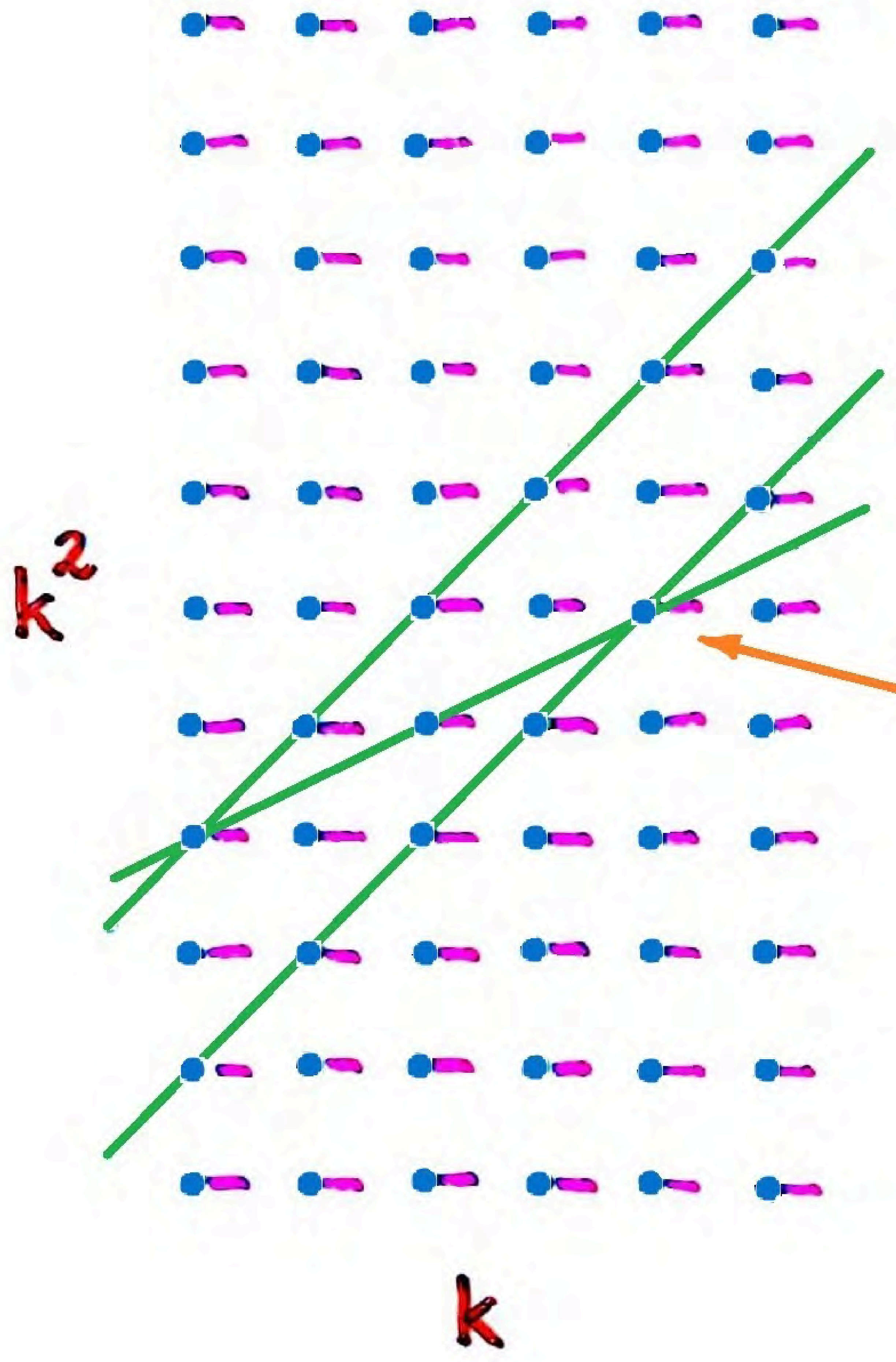

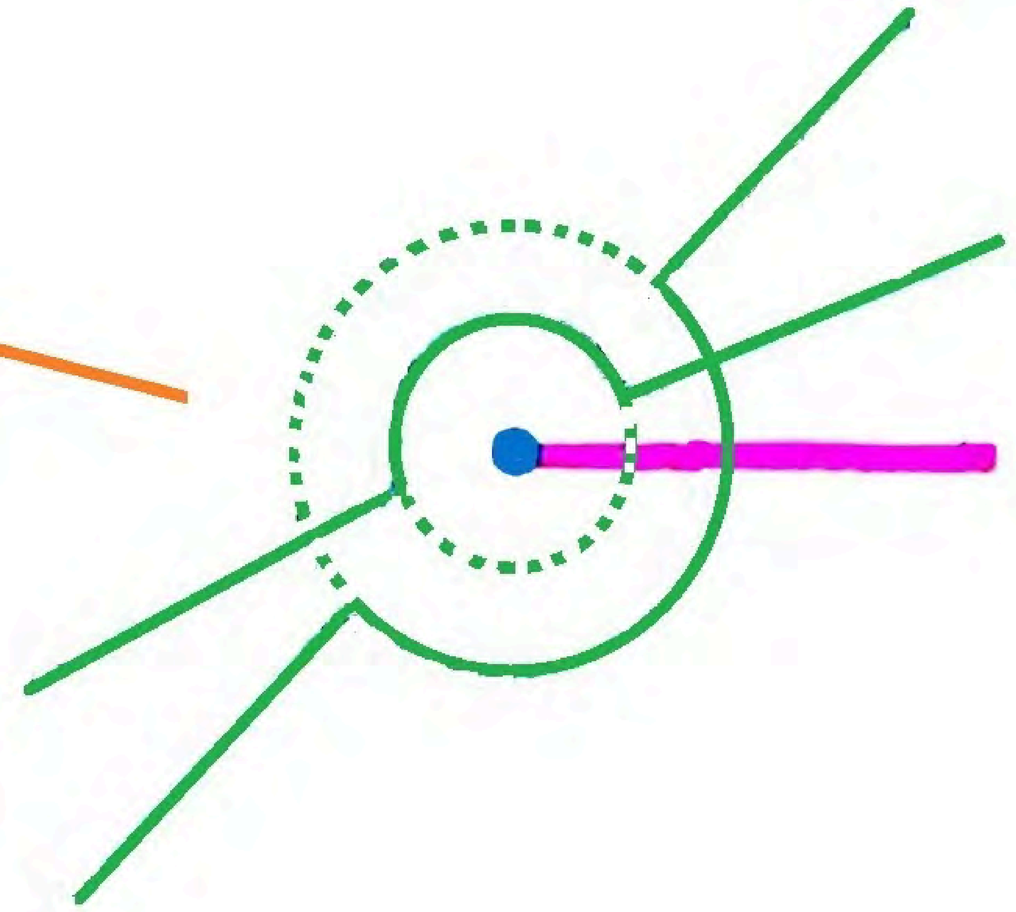

at each incidence

2 choices

A Construction with Pseudosegments

$n = k^3$ short segments

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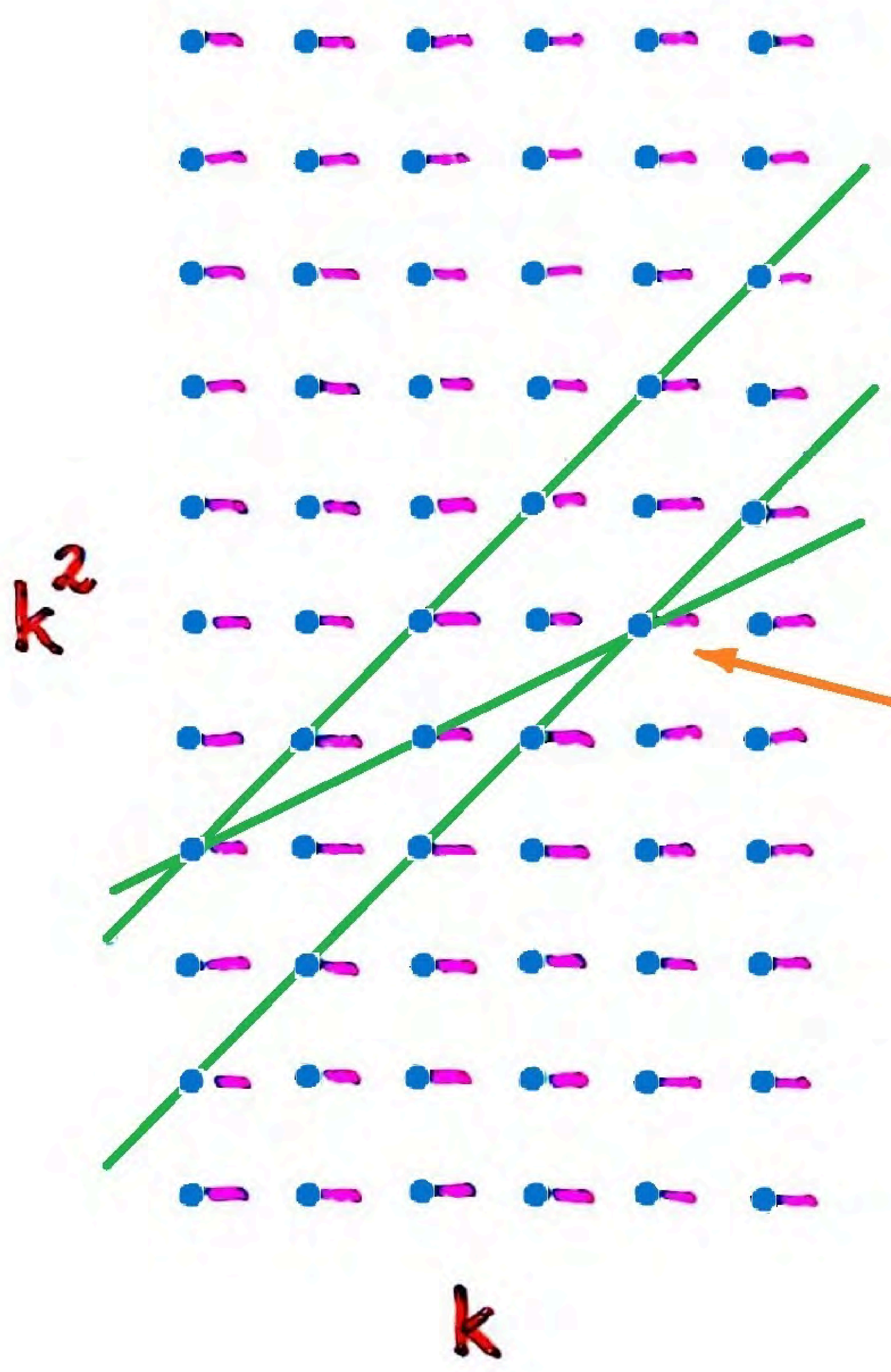
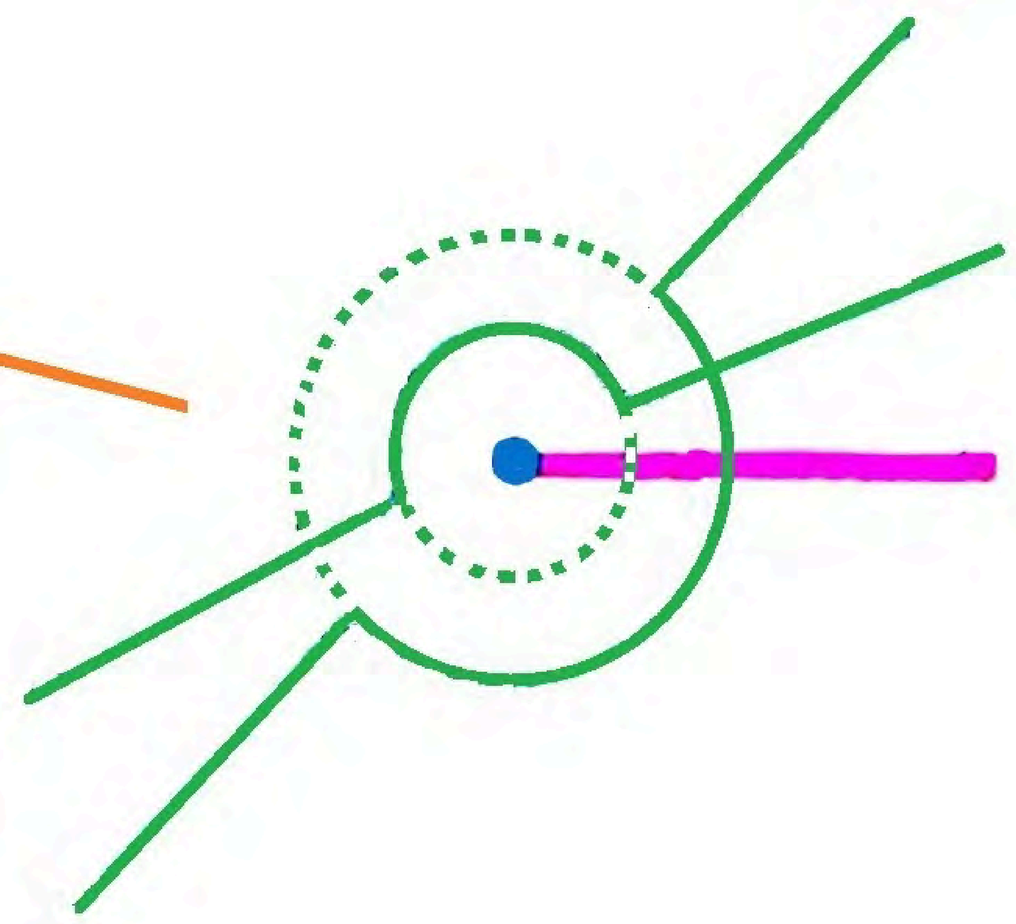




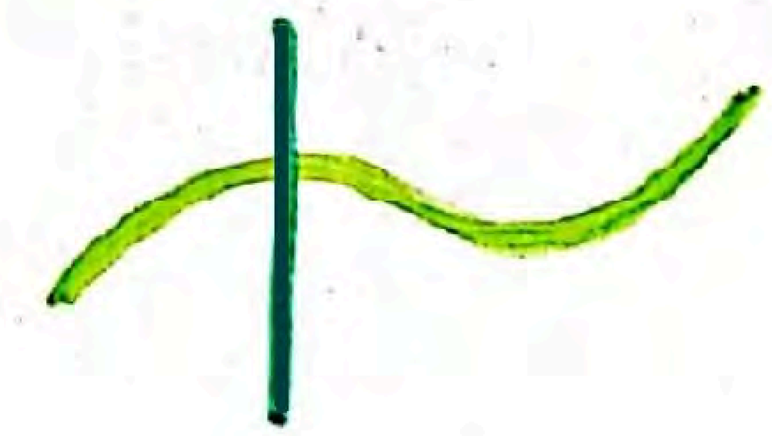
at each incidence

2 choices

$2^{cn^{4/3}}$ different intersection graphs of x-monotone pseudosegments

Enumeration of Pseudosegment Graphs

x-monotone curve - every vertical line intersects it $\leq$ once



Theorem (Fox-P.-Suk 2022)
The number of x-monotone pseudosegments intersection graphs on n labeled vertices, $f(n)$, satisfies

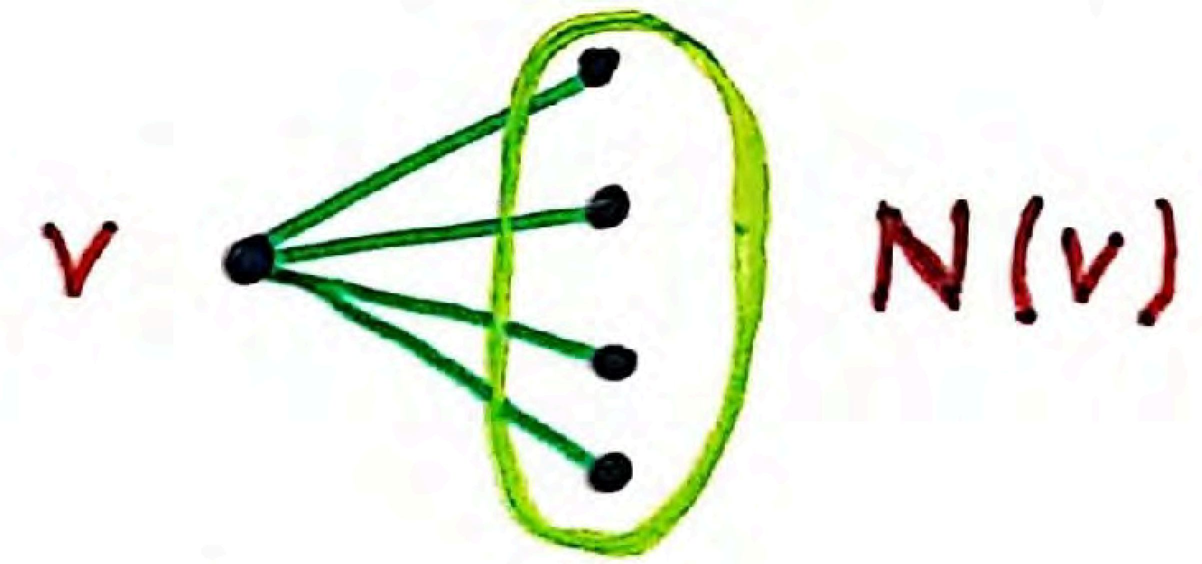
$$2^{\Omega(n^{4/3})} \leq f(n) \leq 2^{O(n^{3/2-\varepsilon})},$$

for some $\varepsilon > 0$.

$2^{O(n^{3/2}\log n)}$ Kynčl 2013

ENUMERATION AND VC-DIMENSION

VC-dimension of a graph = VC-dimension of the family of the neighborhoods of its vertices



ε-perfect partition of the vertex set of G - into K equal parts $V_1 \cup V_2 \cup \dots \cup V_K$ such that for all but $\leq \varepsilon K^2$ pairs of parts

$|E_G(V_i, V_j)| \leq \varepsilon |V_i||V_j|$ or $|E_G(V_i, V_j)| \geq (1-\varepsilon)|V_i||V_j|$.

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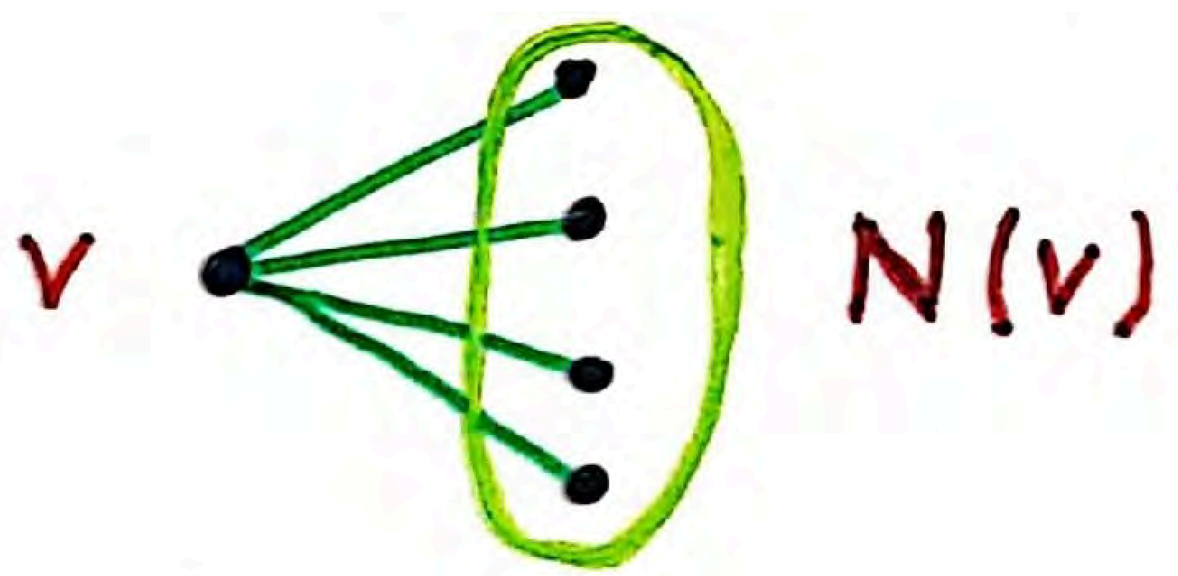


ε-perfect partition of the vertex set of G - into K equal parts $V_1 \cup V_2 \cup \dots \cup V_K$ such that for all but $\leq \varepsilon K^2$ pairs of parts

$|E_G(V_i, V_j)| \leq \varepsilon |V_i||V_j|$ or $|E_G(V_i, V_j)| \geq (1-\varepsilon)|V_i||V_j|$.

Theorem (Lovász-Szegedy 2010, Fox-P.-Suk 2019)
Every graph of VC-dimension d has an ε-perfect partition into $(1/\varepsilon)^{O(d)}$ parts.

bipartite version   Alon-Fischer-Newman 2007

ENUMERATION AND VC-DIMENSION

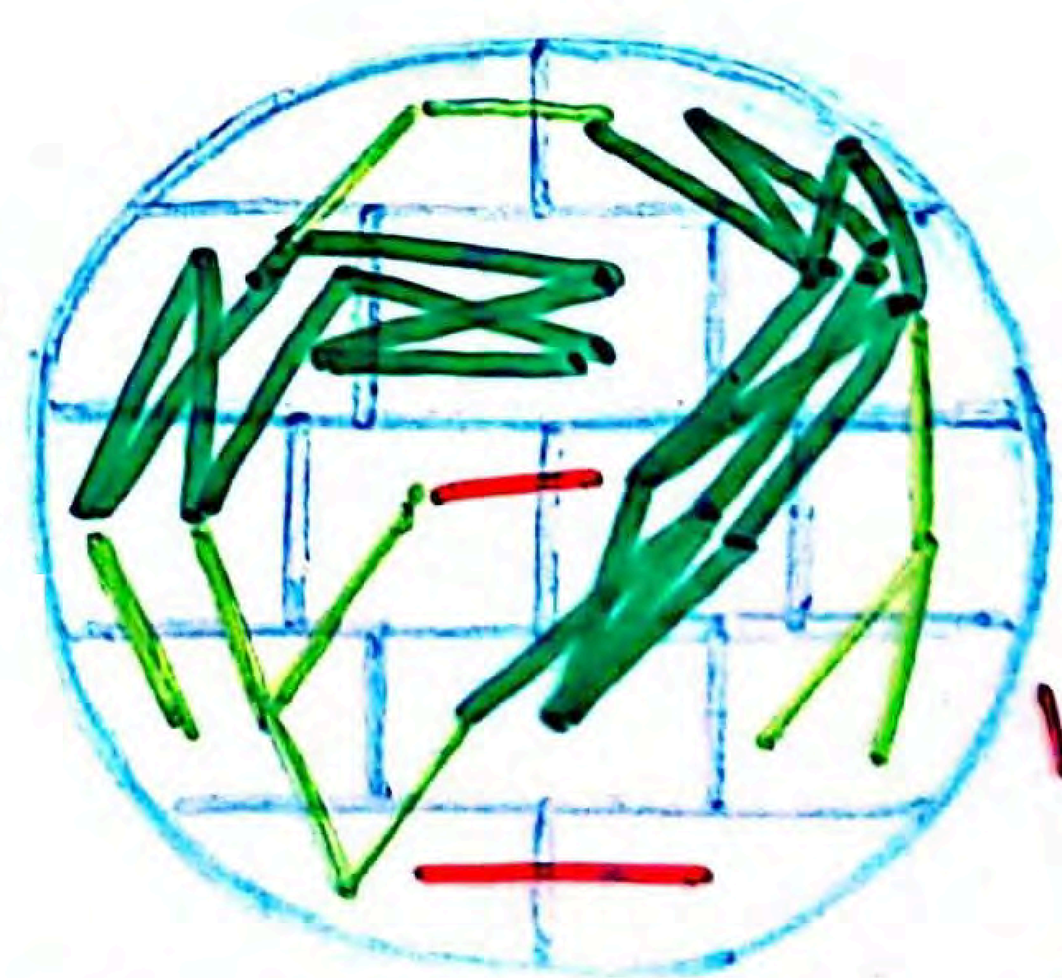
Theorem. Let $\mathcal{G}$ be a hereditary class of graphs. The following statements are equivalent.

(i) The graphs in $\mathcal{G}$ have bounded VC-dimension.

(ii) The number of n-vertex graphs in $\mathcal{G}$ is $2^{o(n^2)}$.

(iii) The graphs in $\mathcal{G}$ admit bounded ε-perfect partitions (i.e., $\mathcal{G}$ satisfies the "ε-perfect regularity lemma").

$|E_G(V_\ell, V_m)| \geq (1-\varepsilon)|V_\ell||V_m|$



$|E_G(V_i, V_j)| \leq \varepsilon |V_i||V_j|$

exceptional