The Heilbronn triangle problem

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joint work with Alex Cohen and Dmitrii Zakharov

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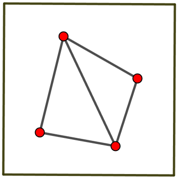

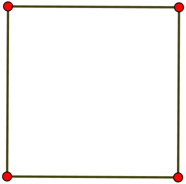

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Construction (Erdős, '50s)

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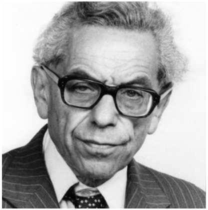

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- For a prime $n \leq p \leq 2n$, define

$$X = \left\{ \left(\frac{x}{p}, \frac{y}{p} \right) : \ x, y \in \{0, \ldots, p-1\}, \ y = x^2 \mod p \right\}.$$

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- This is a set of size p inside $[0,1]^2$ with no three collinear points.
- Any triangle with vertices in X must have area at least $1/2p^2 \geq 1/8n^2$.

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$$\Delta(n) = \Theta\left(\frac{1}{n^2}\right)$$

- In other words, there exist positive absolute constants c and C such that

$$\frac{c}{n^2} \leq \Delta(n) \leq \frac{C}{n^2}.$$

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  - Campos-Jenssen-Michelen-Sahasrabudhe ('24): $d \log d / 2^d$.

Upper bounds

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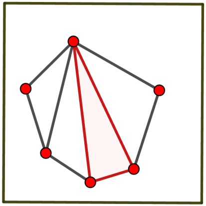

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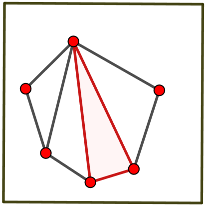

Not easy to improve upon this easy estimate!

Question

Can one find a triangle of area $o(1/n)$?

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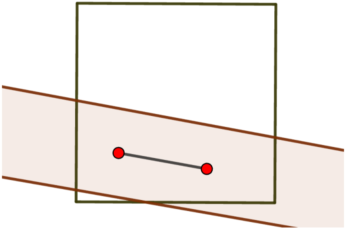

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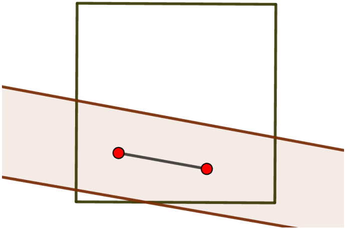

- If Δ denotes the smallest triangle area determined by $P \subset [0,1]^2$, then

$$\mathbb{T}_{x,y}\left(\frac{4\Delta}{\|x-y\|}\right) \cap P = \{x, y\} \quad \text{holds for every } x \neq y \in P.$$

Upper bounds

Theorem (Roth, '51)

$$\Delta(n) = o(1/n).$$

- Density increment argument gives the following quantitative bound:
$$\Delta(n) \lesssim \frac{1}{n(\log\log n)^{1/2}}.$$
- Precursor of Roth's theorem that every set in $\{1, \ldots, n\}$ without nontrivial 3APs must always have size $o(n)$.

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Schmidt shows:

$$\Delta(n) \lesssim \left(\sum_{\{x,y\}\in\binom{P}{2}} \frac{1}{\|x-y\|^2} \right)^{-1/2}$$

Polynomial improvements

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Theorem (Cohen-P.-Zakharov, '24+)

$$\Delta(n) \lessapprox n^{-7/6}.$$

Incidence geometry

Given a set P of points and a set L of geometric objects in $\mathbb{R}^d$, an incidence is a pair $(p, \ell) \in P \times L$, where $p \in P$, $\ell \in L$, and p lies on ℓ.

- We denote by $I(P, L)$ the number of incidences in $P \times L$.
- Szemerédi-Trotter theorem ('83): if $P \subset \mathbb{R}^2$ and L is a set of lines in $\mathbb{R}^2$,

$$I(P, L) \lesssim |P|^{\frac{2}{3}}|L|^{\frac{2}{3}} + |P| + |L|.$$

- Several standard examples showing that this bound is optimal.

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Proving sharp upper bounds for $I(P, L)$ in other settings turns out to lead to remarkably challenging and interesting problems.. (for their own sake and also for most applications)

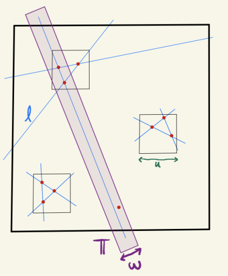
The Heilbronn triangle problem is about incidence lower bounds!

Incidence geometry setup

- $P \subset [0,1]^2$, $L = \{\text{lines } \ell \text{ connecting pairs } x \neq y \in P \text{ with } \|x - y\| \leq u\}$.
- For every scale $w > 0$, let

$$I(w; P, L) = \#\{(p, \ell) \in P \times L \,:\, p \in \mathbb{T}_\ell(w)\}$$

where $\mathbb{T}_\ell(w)$ is the tube of width w generated by ℓ.



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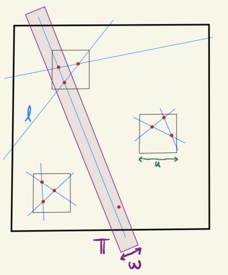


If $I(w; P, L) > 2|L|$ holds for some (tiny) scale w, then $\Delta \lesssim uw$.

Rough story:

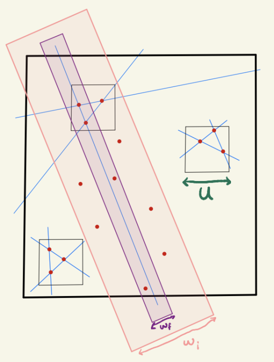
Pick $w_f \ll w_i$:

(A) **Initial estimate.**

$$I(w_i; P, L) \gg w_i |P||L|.$$

(B) **Inductive step.**

$$\left| \frac{I(w_f; P, L)}{w_f |P||L|} - \frac{I(w_i; P, L)}{w_i |P||L|} \right| \ll 1.$$



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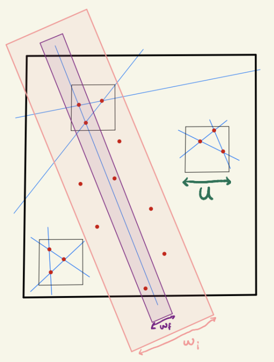

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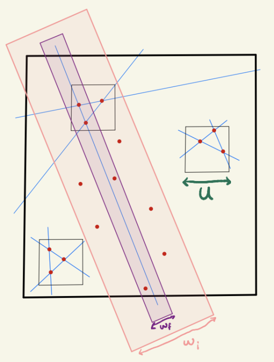

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2. New approach to initial estimate story using direction set estimates from projection theory.
3. New incidence geometry setup (and new combination of steps like 1 and 2).

Inductive step and the high-low method

Incidence setup:

- $P \subset [0,1]^2$, L a set of lines.
- $I(w; P, L) = \#\{(p, \ell) \in P \times L \, : \, p \in \mathbb{T}_\ell(w)\}$.
- Two scales $w_f < w_i$:

$$\left| \frac{I(w_i; P, L)}{w_i |P||L|} - \frac{I(w_f; P, L)}{w_f |P||L|} \right| \lesssim \sqrt{\frac{M_P(w_f)}{|P|} \frac{M_L(w_i)}{|L|} w_f^{-3}}$$

Notation:

- $M_P(w) = \max\{|Q \cap P|, \; Q \text{ a } w \times w \text{ square}\}$,
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Motto

If P and L are not concentrated, then $\frac{I(w;P,L)}{w|P||L|}$ doesn't change much as w varies.

Finite fields

Theorem (Vinh, '11)

Let q be a prime power, let $P \subset \mathbb{F}_q^2$ be a set of points, and let $L \subset \mathbb{F}_q^2$ be a set of lines. Then,

$$\left| I(P, L) - \frac{|P||L|}{q} \right| \leq q^{1/2}|P|^{1/2}|L|^{1/2}.$$

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- Motivation: analogue of Szemerédi-Trotter over $\mathbb{F}_q^2$ when $|P|$ and $|L|$ are large.
- Also comes with a lower bound for $I(P,L)$ when P and L are large, e.g. if $|P||L| > q^3$ then there must always exist an incidence between P and L.
- Original proof uses the expander mixing lemma.

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Cauchy-Schwarz proof:

$$\left| I(P,L) - \frac{|P||L|}{q} \right| = \left| \sum_{x \in P} \left(\psi(x) - \frac{|L|}{q} \right) \right| \leq |P|^{1/2} \left(\sum_{x \in P} \left(\psi(x) - \frac{|L|}{q} \right)^2 \right)^{1/2}.$$

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Show

$$\sum_{x \in P} \left(\psi(x) - \frac{|L|}{q} \right)^2 \leq \sum_{x \in \mathbb{F}_p^2} \left(\psi(x) - \frac{|L|}{q} \right)^2 \leq |L|q.$$

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One can compute explicitly $\sum_{x\in\mathbb{F}_p^2} \psi(x)^2 = \sum_{x\in\mathbb{F}_p^2} \left(\sum_{\ell\in L} 1_{x\in\ell} \right)^2 \ldots$

Main point

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For each $\ell \in L$, let $\Phi_\ell = 1_\ell - \frac{1}{q}$. Then, for $\ell \not\parallel \ell' \in L$:

$$\sum_{x \in \mathbb{F}_q^2} \Phi_\ell(x) \Phi_{\ell'}(x) = 1 - 1 - 1 + 1 = 0.$$

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- $\psi(x) - \frac{|L|}{q} = \sum_{\ell \in L} \Phi_\ell(x)$.
- $\left| I(P,L) - \frac{|P||L|}{q} \right| = \left| \langle \sum_{p \in P} \delta_p, \sum_{\ell \in L} \Phi_\ell \rangle \right|$.
- $\sum_{x \in \mathbb{F}_p^2} \left(\psi(x) - \frac{|L|}{q} \right)^2 = \left\| \sum_{\ell \in L} \Phi_\ell \right\|_2^2 \approx \sum_{\ell \in L} \|\Phi_\ell\|_2^2 = |L| q$.

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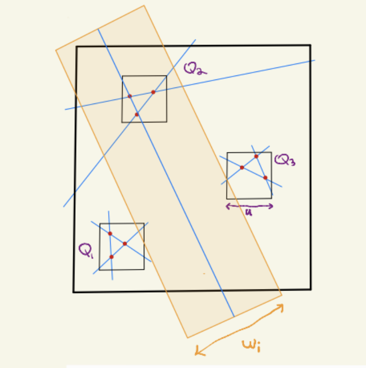
- Estimate $\|\Phi\|_2$ using orthogonality of $\left\{ \frac{1}{w_i} 1_{\mathbb{T}_{w_i}(\ell)} - \frac{1}{w_f} 1_{\mathbb{T}_{w_f}(\ell)} \right\}_{\ell \in L}$.

Initial estimate story

Naive opening moves:

- Partition $[0,1]^2$ into a grid of $u \times u$ squares.
- For each of these squares Q, consider the set of points $P \cap Q$, and define

$$L_Q = \left\{ \ell_\tau \ : \ \tau \in \binom{P \cap Q}{2} \right\}$$



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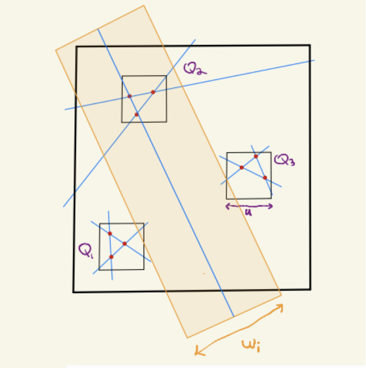

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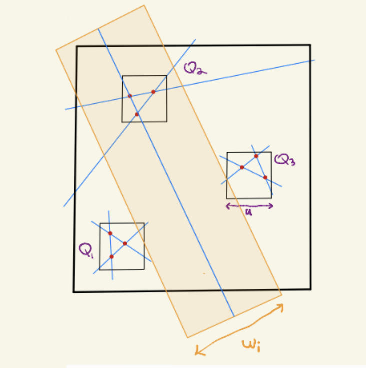

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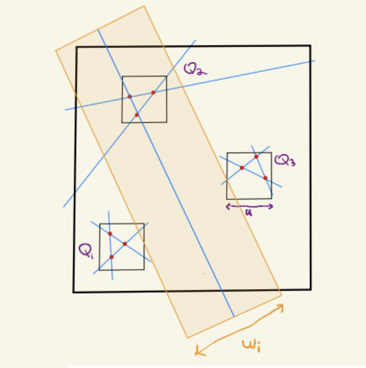
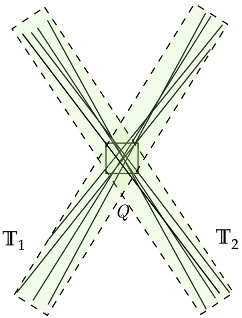
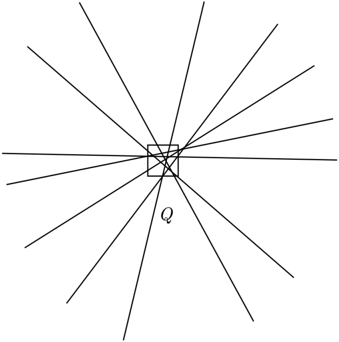


- Using a double counting argument, can ensure that for most of the squares Q in the partition there are not many directions with the $w_i \times 1$ tube in direction θ containing Q having $\lesssim w_i|P|$ points of P.

Main challenge

Address the possibility that the set of directions spanned by the lines in L_Q may be concentrated in the small number of bad directions.





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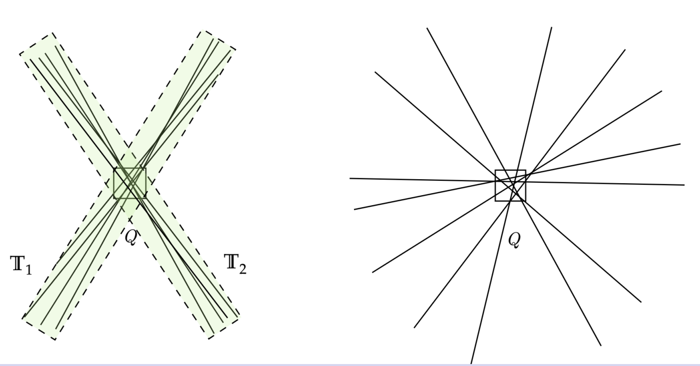

Crucial observation

If $P \cap Q$ is 's-dimensional' for $s > 1$, then this can't really happen.

- Here $\mathcal{P}$ is s-dimensional if $|\mathcal{P} \cap \square| \leq w^s |\mathcal{P}|$ for every $w \times w$ square $\square$.

Projection theory to the rescue

Setup: $S(X) \subset S^1$ denotes the set of directions spanned by a set $X \subset \mathbb{R}^2$.

Theorem (Marstrand '54)

Let $X \subset \mathbb{R}^2$ be a Borel set such that $\dim_H(X) > 1$. Then $\dim_H S(X) = 1$.

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'Twist'

Story so far only recovers $\Delta(n) \lessapprox n^{-8/7}$.

How to find even smaller triangles?

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Crucial observation # 2

If $P \cap Q$ is s-dimensional for $s \leq 1$, then L_Q may not be spread out. But if $P \cap Q$ is not concentrated in a narrow tube, then that can't happen either.

How to find even smaller triangles?

Crucial observation # 2

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Theorem (Orponen, Shmerkin, and Wang, '22)

Let $X \subset \mathbb{R}^2$ be a nonempty Borel set not contained in any line. Then

$$\dim_H S(X) \geq \min\{1, \dim_H X\}.$$

- Continuous analogue of Szőny's theorem that every set $A \subset \mathbb{F}_p^2$ of size $1 < |A| \leq p$ determines at least $\frac{|A|+3}{2}$ distinct directions, provided that A is not contained in any affine line.
- Proof relies on Bourgain's discretized sum-product theorem.

Discretize $\Longrightarrow$ Incorporate $(1-\epsilon)$-regular sets $\Longrightarrow$ Better initial estimate.

New incidence geometry setup

New incidence geometry setup

Question

For $\delta > 0$, what is the maximum size of $n = n(\delta)$ for which there exists a set of points $P = \{p_1, \ldots, p_n\} \subset [0,1]^2$ and a set of lines $L = \{\ell_1, \ldots, \ell_n\}$ such that

$$p_i \in \mathbb{T}_{\ell_j}(\delta) \text{ if and only if } i = j?$$

Here $T_\ell(\delta)$ denotes the tube of width δ centered around line ℓ.

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Easy bound

$$n \leq 1/\delta^2$$

- For every set $P \subset [0,1]^2$, there exists $i \neq j$ such that $|p_i - p_j| < 1/n^{1/2}$.
- If $n > 1/\delta^2$, then $|p_i - p_j| < 1/n^{1/2} < \delta$ holds, and so the point p_j lies in the tube $\mathbb{T}_{\ell_i}(\delta)$.

Can one do better?

Finite fields analogue

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Let q be a prime power, let $P = \{p_1, \ldots, p_n\} \subset \mathbb{F}_q^2$, and let $L = \{\ell_1, \ldots, \ell_n\}$ be a set of lines in $\mathbb{F}_q^2$ with

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What is the maximum value of n?

The lower bound in Vinh's inequality gives

$$n = I(P, L) \geq \frac{1}{q}|P||L| - q^{1/2}|P|^{1/2}|L|^{1/2} = \frac{n^2}{q} - q^{1/2}n.$$

Hence $n \leq q^{3/2} + q$.

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Let $P \subset [0,1]^2$. To find a triangle of area $\lessapprox n^{-7/6}$ determined by P:

- Pick $m \gtrsim n$ disjoint pairs $(p_1, q_1), \ldots, (p_m, q_m) \in P \times P$ such that $|p_i - q_i| < 1/n^{1/2}$ for each i.
- Let ℓ_i denote the line passing through p_i and q_i and consider the set $P' = \{p_1, \ldots, p_m\}$ and the lines $L = \{\ell_1, \ldots, \ell_m\}$.
- There is some $i \neq j$ for which $d(p_i, \ell_j) \lessapprox n^{-2/3}$.
- Triangle $p_i p_j q_j$ has area $\lessapprox n^{-2/3} n^{-1/2} = n^{-7/6}$.

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Problem

If q is a prime number, then $n \leq q^{3/2-c}$ holds for some absolute constant $c > 0$.

Thank you